Homogenization of the Hyperbolic Problems with Imperfect Interfaces

Yang Zhanying¹,² and Yu Yunxia²

(¹ College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China; ² Department of Mathematics, Xinxiang University, Xinxiang 453000, China)

Abstract In this paper we study a class of hyperbolic problem with non-periodic coefficients in a two-component domain. By the time-dependent periodic unfolding method in two-component domains we derive the homogenization and corrector results which generalize those for the case with periodic coefficients.

Keywords hyperbolic problems; periodic unfolding method; homogenization; correctors

1 Introduction

In this paper we study the homogenization and corrector results for the following problem with $\gamma < -1$:

$$\begin{align*}
\left\{ \begin{array}{ll}
\varepsilon u_{1x} - div (A^e \nabla u_{1e}) &= f_{1e} \text{ in } \Omega_{e1} \times (0,T) \\
u_{1e} &= 0\text{ on } \partial \Omega \times (0,T) \\
u_{2e} &= 0\text{ on } \partial \Omega \times (0,T) \\
A^e \nabla u_{1e} \cdot n_{1e} &= -A^e \nabla u_{2e} \cdot n_{2e}\text{ on } \Gamma^x \times (0,T) \\
A^e \nabla u_{1e} \cdot n_{1e} &= -\varepsilon h^x (u_{1e} - u_{2e})\text{ on } \Gamma^x \times (0,T) \\
u_{1n} &= 0\text{ on } \partial \Omega \times (0,T) \\
u_{2n} &= 0\text{ on } \partial \Omega \times (0,T) \\
u_{1e} (x,0) &= U^1_{1e} (x) = U^1_{1e} \text{ on } \Omega_{e1} \\
u_{2e} (x,0) &= U^2_{1e} (x) = U^2_{1e} \text{ on } \Omega_{e2} \\
\end{array} \right.
\end{align*}$$

(1)

where $\Omega \subset \mathbb{R}^n$ is a domain which is the union of two $\varepsilon$-periodic sub-domains $\Omega_{e1}$ and $\Omega_{e2}$ separated by an interface $\Gamma^x$ such that $\Omega_{e1} \cup \Omega_{e2} = \Omega$ and $\Gamma^x = \partial \Omega_{e2}$.

Here $\Omega_{e1}$ is connected and the number of connected components of $\Omega_{e2}$ is of order $\varepsilon \to 0$. This problem models the wave propagation in a medium made up of two materials with different coefficients of propagation.

For the physical model we refer the reader to Carslaw and Jaeger⁰.

Let $Y = [0,1] \times [0,1] \times [0,1]$ be the reference cell with $l_{i} > 0$ for all $i = 1, \ldots, n$. Throughout the paper we have the following assumptions:

For any $\varepsilon \Omega A = (a_{ij}^x)_{1 \leq i,j \leq n}$ is a matrix satisfying the following:

$$A^e$$ is symmetric and there exist $\alpha \beta \in \mathbb{R}^+$ such that $(A^e \lambda) \geq \alpha \lambda$ for all $\lambda \in \mathbb{R}^+$ and $a.e. x \in \Omega$.

For any $\varepsilon h^x (x) = h (x/\varepsilon)$ where $h$ is a $Y$-periodic function such that $h \in L^\infty (\Gamma)$ and there exists
such that $0 < h_0 < h(y)$ a.e. on $\Gamma$.

The initial data satisfy the following assumptions:

\[
U_0^\varepsilon = (U_0^1\varepsilon \ldots U_0^n\varepsilon) \in V \times H^1(\Omega) \quad \text{and} \quad f_\varepsilon = (f_1\varepsilon \ldots f_n\varepsilon) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))
\]

For the classical case $A^*(x) = A(x/\varepsilon)$ with $A$ being periodic, symmetric, bounded and uniformly elliptic, Donato, Faella and Monsurro gave the homogenization of problem (1) for $\gamma \leq 1$ in Ref. [3]. Subsequently, they proved the corrector results in Ref. [4] for $-1 < \gamma \leq 1$. Their proofs are based on the oscillating test functions method. For $\gamma < -1$, the first author recently gave the corrector results by the unfolding method in Ref. [5]. But these methods do not work for the case that $A^*(x)$ is non-periodic coefficient matrix.

In this paper, we will consider problem (1) with $A^*(x)$ being non-periodic for $\gamma < -1$. More precisely, suppose that there exists a matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ such that:

\[
\mathcal{F}(A^*) \rightarrow A \text{ strongly in } (L^1(\Omega \times Y))^n
\]

where $\mathcal{F}$ is the unfolding operator. By the unfolding method in two-component domains, we derive the homogenization and corrector results for $\gamma < -1$. Next, we state our main theorems in which we will use some notations to be defined in the next section.

We first state the homogenization result which recover those in Ref. [2].

**Theorem 1** For $\gamma < -1$, let $u_\varepsilon$ be the solution of problem (1) with (2). We further suppose that:

\[
\|U_0^\varepsilon\|_{L^1} \quad \text{is uniformly bounded.}
\]

\[
U_\varepsilon \rightarrow (\theta_1 U_\varepsilon^1 \theta_2 U_\varepsilon^2) \text{ weakly in } L^2(\Omega) \times L^2(\Omega)
\]

where $U_\varepsilon^1 \in H_0^1(\Omega) \rightarrow \theta_1 U_\varepsilon^1 \theta_2 U_\varepsilon^2$ weakly in $L^2(\Omega) \times L^2(\Omega)$ and $f_\varepsilon \rightarrow -\theta f_\varepsilon |_{\Gamma^0} \theta f_\varepsilon^2$ weakly in $L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$.

Then there exists $U_1 \in L^s(0, T; H_0^1(\Omega))$ such that:

\[
u_\varepsilon \rightarrow \theta U_1 \text{ weakly }^* \text{ in } L^2(0, T; L^2(\Omega)) |_{\Gamma^0} \leq 1 \|2
\]

Also $U_1$ is the unique solution of the following problem:

\[
u_1 = -\text{div}(A^* \nabla u_1) + \theta f_1 + \theta f_2 \quad \text{in } \Omega \times (0, T)
\]

\[
u_1 = 0 \quad \text{on } \partial \Omega \times (0, T)
\]

\[
u_1(x, 0) = \theta_1 U_0^1 + \theta_2 U_0^2 \quad \text{in } \Omega
\]

\[
u_1(x, T) = \theta_1 U_1^1 + \theta_2 U_1^2 \quad \text{in } \Omega
\]

where the homogenized matrix $A^0 = (a_{ij}^0(x))_{1 \leq i, j \leq n}$ is defined by:

\[
a_{ij}^0(x) = M_i(a_{ij} + \sum_{k=1}^n \partial_k a_{ij} \frac{\partial Y_j}{\partial Y_k})
\]

and $Y_j \in L^s(\Omega; H^{1, p}(Y))$ satisfies $M_i(Y_j) = 0$ in $\Omega$. Further, we have the following precise convergence of flux:

\[
u_\varepsilon \nu_\varepsilon \rightarrow A_1 \nabla U_1 \text{ weakly }^* \text{ in } L^s(0, T; L^2(\Omega))
\]

\[
u_\varepsilon \nu_\varepsilon \rightarrow A_1 \nabla U_1 \text{ weakly }^* \text{ in } L^s(0, T; L^2(\Omega))
\]

where $A_i = (a_{ij}^0(x))_{1 \leq i, j \leq n}$. In Section 3, we prove the homogenization result and weakly* convergence of the homogenized matrix $A^0$ still depends on $x$.

In order to investigate the corrector results, we need stronger assumptions on the initial data than that of the convergence results as already evidenced in the classical works. Here, we impose the following assumptions as introduced by the first author in Ref. [4] which are slightly weaker than those in Ref. [3].

(i) For $f_\varepsilon \in L^2(0, T; L^2(\Omega), \omega) (i = 1, 2)$ there exists $f_i \in L^2(0, T; L^2(\Omega))$ such that:

\[
\|f_\varepsilon - f_i\|_{L^2(0, T; L^2(\Omega), \omega)} \to 0 \quad \text{for } i = 1, 2.
\]

(ii) For $U_\varepsilon \in L^2(\Omega, \omega)$ there exists $U^i \in L^2(\Omega)$ such that:

\[
\|U_\varepsilon^i - U^i\| \to 0.
\]

(iii) For $U_\varepsilon$ we assume that:
\[ \| U^0 \|_H^2 \text{ is uniformly bounded} \]
\[ \hat{U}^{\gamma}_{\infty} \rightharpoonup U^0 \text{ weakly in } L^2(\Omega) \]
\[ \int_{\partial\Omega} A^\varepsilon \nabla u_{1\varepsilon} \cdot \nabla u_{1\varepsilon} \, dx + \ldots \]

where \( U^0 \in H^2_0(\Omega) \).

Under these assumptions \( U \) converges to that of the homogenized one. Moreover we obtain Theorem 2.

**Theorem 2** For \( \gamma < -1 \) let \( u_\varepsilon \) be the solution of problem (1) with (2). Suppose that the initial data satisfy (5) \( \sim \) (7). Let \( u_1 \) be the solution of the homogenized problem (3) \( \in \) then we have the following corrector results:

\[ \begin{align*}
\| u_{1\varepsilon} - u_{2\varepsilon} - u_1 \|_{L^2(\Omega)} & \to 0 \\
\| \nabla u_{1\varepsilon} - \nabla u_1 - \sum_{i=1}^n U^i_1(\frac{\partial u_1}{\partial \nu_i})(\nabla \chi_i) \|_{L^2(\Omega \setminus \Gamma(\Omega))} & \to 0 \\
\| \nabla u_{2\varepsilon} - \nabla u_1 - \sum_{i=1}^n U^i_2(\frac{\partial u_1}{\partial \nu_i})U^i_2(\nabla \chi_i) \|_{L^2(\Omega \setminus \Gamma(\Omega))} & \to 0
\end{align*} \]

where \( \chi_i \in L^\infty(\Omega; H^1_0(Y)) \) \( \in \) is the solution of the cell problem (4).

For the parabolic case the homogenization for \( \gamma \leq 1 \) and the corrector results for \( -1 < \gamma \leq 1 \) were given in Ref. [11] and [16] respectively. Recently by the unfolding method the first author gave the homogenization and corrector results for \( \gamma \leq 1 \). Our results are also related to those of hyperbolic problems in perforated domains in Ref. [7] and [8].

This paper is organized as follows. In Section 2 we briefly recall some properties related to the unfolding method in two-component domains. Section 3 is devoted to the homogenization results. In Section 4 we prove the corrector results.

## 2 Preliminaries

In this section we briefly recall some results about the unfolding method. The periodic unfolding method was originally introduced in Ref. [9]. Then it was extended to perforated domains in Ref. [10].

### 2.1 Some notations

Let \( \Omega \subseteq \mathbb{R}^d \) be an open and bounded set with Lipschitz continuous boundary. We suppose that \( Y_1 \) and \( Y_2 \) are two nonempty open disjoint subsets of \( Y \) such that \( Y = Y_1 \cup Y_2 \) where \( Y_1 \) is connected and \( \Gamma \equiv \partial Y_2 \) is Lipschitz continuous.

Let \( \varepsilon \) be the general term of a sequence of positive real numbers which converges to zero. For any \( k \in \mathbb{Z}^d \) we denote:

\[ Y^k_i = k_i + Y \setminus \Gamma^k_i = k_i + Y \]

where \( k_i = (k_i l_1 \cdots l_{d-1}) \) and \( l_k = 1 \). For any fixed \( \varepsilon \) let \( K_\varepsilon = \{ k \in \mathbb{Z}^d \mid \varepsilon Y^k_i \cap \Omega \neq \emptyset \} \). We suppose that \( \partial \Omega \cap \bigcup_{k \not\in K_\varepsilon} (\varepsilon Y^k_i) \) is \( \emptyset \) and define the two components of \( \Omega \) and the interface respectively by:

\[ \Omega_{\varepsilon} = \bigcap_{k \not\in K_\varepsilon} \varepsilon Y^k_i \cap \Omega = \Omega \setminus \partial \Omega_{\varepsilon} \]

Observe that \( \partial \Omega \) and \( \Gamma^\varepsilon \) are disjoint and the component \( \Omega_{\varepsilon} \) is union of \( \varepsilon \)-disjoint translated sets of \( \varepsilon Y^k_i \).

Now we introduce two spaces \( V^\varepsilon \) and \( H^2_{\gamma} \). Define \( V^\varepsilon \) by:

\[ V^\varepsilon = \{ v \in H^1(\Omega_{\varepsilon}) \mid v = 0 \text{ on } \partial \Omega \} \]

endowed with the norm \( \| v \|_{V^\varepsilon} = \| \nabla v \|_{L^2(\Omega_{\varepsilon})} \). For any \( \gamma \in \mathbb{R} \) the product space

\[ H^2_{\gamma} = \{ u = (u_1, u_2) \mid u_i \in V^\varepsilon \cap H^2(\Omega_{\varepsilon}) \} \]

is equipped with the norm:

\[ \| u \|_{H^2_{\gamma}}^2 = \| \nabla u_1 \|_{H^2(\Omega_{\varepsilon})}^2 + \| \nabla u_2 \|_{H^2(\Omega_{\varepsilon})}^2 + \varepsilon^2 \| u_1 - u_2 \|_{L^2(\Gamma^\varepsilon)}^2 \]

The following notations are related to the unfolding method (see also Ref. [11]):

\[ K_\varepsilon = \{ k \in \mathbb{Z}^d \mid \varepsilon Y^k \subseteq \Omega \} \]

\[ \Omega_{\varepsilon} = \bigcap_{k \not\in K_\varepsilon} \varepsilon Y^k \cap \Omega = \Omega \setminus \partial \Omega_{\varepsilon} \]

\[ \Omega_{\varepsilon} = \bigcup_{k \not\in K_\varepsilon} \varepsilon Y^k \cap \Omega = \Omega \setminus \partial \Omega_{\varepsilon} \]

In what follows we will use the following notations:

\[ \theta_i = 1 \mid Y_i \mid / \mid Y \mid = 1 \]

\[ M_{\varepsilon}(v) = \frac{1}{\theta_i} \int_{Y_i} v \, dx \]
$g$ is the zero extension to $\Omega$ (respectively $\Omega \times A$) of any function $g$ defined on $\Omega_i \varepsilon$ (respectively $\Omega_i \varepsilon \times A$) for $i = 1 \mathbb{P}$.

$C$ denotes generic constant which does not depend upon $\varepsilon$.

The notation $L^p(\Omega)$ will be used both for scalar and vector-valued functions defined on the set $\Omega$ since no ambiguity will arise.

2.2 Some properties

In this subsection we briefly recall some results related to the time-dependent unfolding operator in two-component domains. We refer the reader to Ref. [4-14] and [11-14] for further properties and related comments.

**Proposition 1** Let $i = 1 \mathbb{P}$. For $p \in [1, \infty)$ and $q \in [1, \infty]$ let $\phi \in L^p(0T; L^1(\Omega_{\varepsilon}))$. For a.e. $t \in (0T)$ we have:

$$\frac{1}{|Y_i|} \int_{Y_i \times t} T_i^\varepsilon(\phi)(x, y) \, dx \, dy =$$

$$\int_{\partial \Omega_i} \phi(x) \, dx - \int_{\partial \Omega_i} \phi(x) \, dx - \int_{\Omega_{\varepsilon}} \phi(x) \, dx.

**Proposition 2** Let $\Omega \in [1, \infty)$. For $i = 1 \mathbb{P}$

(i) Let $w \in L^1(0T; L^1(\Omega))$ then we have:

$$\| T_i^\varepsilon(\phi)(x, y) - w \|_{L^1(0T; L^1(\Omega_i \varepsilon))} \rightarrow 0.$$

(ii) Let $w_i \in L^1(0T; L^1(\Omega_{\varepsilon}))$ and $w \in L^1(0T; L^1(\Omega \varepsilon))$ then the following two assertions are equivalent:

(a) $T_i^\varepsilon(\phi)(x, y) \rightarrow w$ strongly in $L^1(0T; L^1(\Omega \varepsilon))$ and $w = \lim_{t \rightarrow 0} \phi(x, y) - w \rightarrow 0$

(b) $\| w_i - w \|_{L^1(0T; L^1(\Omega_{\varepsilon}))} \rightarrow 0$.

(iii) Let $w_i \in L^1(0T; L^1(\Omega_{\varepsilon}))$ and $w \in L^1(0T; L^1(\Omega \times Y_i))$ then the following two assertions are equivalent:

(a) $T_i^\varepsilon(\phi)(x, y) \rightarrow w$ strongly in $L^1(0T; L^1(\Omega \times Y_i))$ and $w = \lim_{t \rightarrow 0} \phi(x, y) - w \rightarrow 0$

(b) $\| w_i - U_i^\varepsilon(\phi) \|_{L^1(0T; L^1(\Omega_{\varepsilon}))} \rightarrow 0$.

**Proposition 3** Let $p \mathbb{P}$ and $T_i^\varepsilon(\phi)(x, y) \rightarrow 0$ for $i = 1 \mathbb{P}$. Let $f \in L^q(0T; L^1(\Omega))$ and $g \in L^1(\Omega; L^1(\Omega \times Y_i))$. Then we have:

$$\| U_i^\varepsilon(fg) - U_i^\varepsilon(f) U_i^\varepsilon(g) \|_{L^2(0T; L^2(\Omega_i \varepsilon))} \rightarrow 0.$$

We end this subsection with the following convergence theorem which is crucial to obtaining our homogenization results.

**Theorem 1** Let $u_\varepsilon = \{ u_{i\varepsilon} \}_{\varepsilon \rightarrow 0}$ and $\{ u_\varepsilon \}_{\varepsilon \rightarrow 0}$ be a sequence in $L^2(0T; L^2(\Omega \varepsilon))$ with $\varepsilon \rightarrow 0$. If

$$\| u_\varepsilon \|_{L^2(T; L^2(\Omega \varepsilon))} + \| u_\varepsilon \|_{L^2(T; L^2(\Omega \varepsilon))} \leq C$$

then there exist $u \in L^2(0T; H^1_0(\Omega))$ and $w \in L^2(\Omega; H^1(\Omega \varepsilon))$ such that (i) up to a subsequence (still denoted by $\varepsilon$)

(i) $\tilde{T}_i^\varepsilon(u_{i\varepsilon}) \rightarrow u_\varepsilon$ strongly in $L^2(0T; L^2(\Omega \varepsilon))$.

(ii) $\tilde{T}_i^\varepsilon(u_{i\varepsilon}) \rightarrow u_\varepsilon$ weakly in $L^2(0T; L^2(\Omega))$.

(iii) $\tilde{T}_i^\varepsilon(\nabla u_{i\varepsilon}) \rightarrow \nabla u_\varepsilon + \nabla \tilde{u}_\varepsilon$ weakly in $L^2(0T; L^2(\Omega \times Y_i))$.

(iv) $\tilde{T}_i^\varepsilon(u_{i\varepsilon}) \rightarrow u_\varepsilon$ weakly in $L^2(0T; L^2(\Omega \varepsilon))$.

(v) $\tilde{T}_i^\varepsilon(\nabla u_{i\varepsilon}) \rightarrow \nabla u_\varepsilon$ weakly in $L^2(0T; L^2(Y_i))$.

where $\tilde{T}_i^\varepsilon(\tilde{u}_\varepsilon) = 0(1 = i \mathbb{P})$ for a.e. $x \in \Omega$.

Moreover

$$\tilde{u}_\varepsilon = \tilde{u}_\varepsilon - y_\varepsilon \nabla u_{i\varepsilon} \text{ on } [0, T] \times X_i \times Y_i$$

where $y_\varepsilon = y - \varepsilon \mathbb{F}(y) \varepsilon$.

3 Homogenization result

For every fixed $\varepsilon \mathbb{P}$ the classical arguments provides that problem (1) has a unique solution $u_\varepsilon$ satisfying the following uniform estimate:

$$\| u_\varepsilon \|_{L^2(0T; L^2(\Omega \varepsilon))} + \| u_\varepsilon \|_{L^2(0T; L^2(\Omega \varepsilon \times Y_i))} \leq C.$$

Now we state the unfolded formulation of the homogenization results (see Theorem 1) which will be
used for getting the corrector results.

**Theorem 4** Under the same hypotheses as in Theorem 1 there exist \( u_1 \in L^\infty(\Omega; H^1_0(\Omega)) \) with \( u'_1 \in L^\infty(0; T; L^2(\Omega; H^1_0(\Omega))) \) and \( \tilde{u}_2 \in L^\infty(0; T; L^2(\Omega; H^1_0(Y, \gamma))) \) such that:

(i) \( \mathcal{T}_1'(u_{1\varepsilon}) \to u_1 \) strongly in \( L^\infty(0; T; L^2(\Omega; H^1(Y, \gamma))) \) for any \( \varepsilon \in (1, +\infty) \)

(ii) \( \mathcal{T}_1'(u_{1\varepsilon}) \to u_1 \) weakly in \( L^\infty(0; T; L^2(\Omega; H^1(Y, \gamma))) \)

(iii) \( \mathcal{T}_2'(u_{2\varepsilon}) \to \nabla u_1 + \nabla \hat{u}_2 \) weakly in \( L^\infty(0; T; \nabla \Omega; Y, \gamma)) \)

(iv) \( \mathcal{T}_2'(u_{2\varepsilon}) \to u_1 \) weakly in \( L^\infty(0; T; L^2(\Omega; H^1(Y, \gamma))) \)

(v) \( \mathcal{T}_2'(u_{2\varepsilon}) \to \nabla \hat{u}_2 \) weakly in \( L^\infty(0; T; \nabla \Omega; Y, \gamma)) \)

(vi) \( \mathcal{T}_2'(u_{2\varepsilon}) \to u_1 \) weakly in \( L^\infty(0; T; L^2(\Omega; H^1(Y, \gamma))) \)

where \( \mathcal{T}_1' \) denotes the corrector results. Firstly we impose the assumptions (5) \( \sim (7) \) as presented in Ref. [4] which are slightly weaker than those in Ref. [3]. Under these assumptions the energy of problem (1) converges in \( C^0(\Omega; 0; T; \Omega)) \) to that of the homogenized one. Moreover we obtain that some convergences in Theorem 4 are strong ones.

Now we focus on the energy of the problem (1). For each \( \varepsilon \) the energy \( E'(t) \) is defined by:

\[
E'(t) := \frac{1}{2} \int_{D_{\varepsilon}} u_{1\varepsilon} \cdot \nabla u_{1\varepsilon} dx + \frac{1}{2} \int_{D_{\varepsilon}} \nabla u_{1\varepsilon} \cdot \nabla u_{1\varepsilon} dx + \int_{D_{\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \cdot \nabla u_{1\varepsilon} dx + \int_{D_{\varepsilon}} \sigma^\varepsilon \nabla u_{1\varepsilon} \cdot \nabla u_{1\varepsilon} dx
\]

The energy associated to the homogenized problem (3) is defined by:

\[
E(t) := \frac{1}{2} \int_{D} u_1 \cdot \nabla u_1 dx + \int_{D} A^0 \nabla u_1 \cdot \nabla u_1 dx.
\]

Following the standard framework we can get the following convergence of energy.

**Theorem 5** Let \( \gamma \leq -1 \). Suppose that \( u_{1\varepsilon} \) is the solution of problem (1) with the initial data satisfying (5) \( \sim (7) \). Let \( u_1 \) be the solution of the homogenized
problem (3) then we have:
\[ E^*(t) \rightarrow E(t) \text{ strongly in } C^0(\Omega(0,T)). \]

**Corollary 1** Under the assumptions of Theorem 5, we have:
1. \[ \| u'_{i}\|_{L^2(\Omega(0,T);L^2(\Omega \times Y))} \rightarrow 0 \]
2. \[ \nabla u_{i} \rightarrow \nabla \hat{u}_{i} \text{ strongly in } L^2(0;L^2(\Omega \times Y)) \]
3. \[ \mathcal{F}^{*}_{\varepsilon}(\nabla u_{i}) \rightarrow \nabla \hat{u}_{i} \text{ strongly in } L^2(0;L^2(\Omega \times Y)) \]

for \( i = 1,2 \).

Using Proposition 1 we obtain:
\[ \lim_{\varepsilon \rightarrow 0} \int_{0}^{T} E'(t) dt = \liminf_{\varepsilon \rightarrow 0} \int_{0}^{T} E'(t) dt \leq \limsup_{\varepsilon \rightarrow 0} \int_{0}^{T} E'(t) dt \leq \int_{0}^{T} E(t) dt \]

where
\[ E^*(t) : = \frac{1}{2} \int_{\Omega} | u'_{i} |^2 dx + \int_{\Omega} | u'_{2,\varepsilon} |^2 dx + \int_{\Omega} A^* \nabla u_{1} \nabla u_{1} dx + \int_{\Omega} A^* \nabla u_{2,\varepsilon} \nabla u_{2,\varepsilon} dx. \]

According to the ellipticity of \( A^* \), these give the first line in (i) of Corollary 1.

The latter equality in Equation (11) indicates:
\[ \int_{0}^{T} \int_{\Omega \times Y} \mathcal{F}^{*}_{\varepsilon}(u_{i}') dx dy dt + \int_{0}^{T} \int_{\Omega \times Y} \mathcal{F}^{*}_{\varepsilon}(u_{2,\varepsilon}') \sqrt{\varepsilon} dx dy dt = \int_{0}^{T} \int_{\Omega \times Y} A(\nabla u_{1}') \mathcal{F}^{*}_{\varepsilon}(\nabla u_{1}) \mathcal{F}^{*}_{\varepsilon}(\nabla u_{1}) dx dy dt + \int_{0}^{T} \int_{\Omega \times Y} A(\nabla u_{2,\varepsilon}') \mathcal{F}^{*}_{\varepsilon}(\nabla u_{2,\varepsilon}) \mathcal{F}^{*}_{\varepsilon}(\nabla u_{2,\varepsilon}) dx dy dt \rightarrow 2 \int Y \int_{0}^{T} E(t) dt. \]
Proof of Theorem 2 Notice that $u_1$ is independent of $y$. By (ii) of Proposition 2, the first convergence in (8) follows from (i) in Corollary 1. By (i)-(iii) in Corollary 1 we use (iii) of Proposition 2 to get:

\[
\| \nabla u_{1e} - U_1^e( \nabla u_1 + \nabla \tilde{u}_1) \| \leq L(\Omega',\Omega',\Omega') \rightarrow 0
\]

\[
\| \nabla u_{2e} - U_2^e( \nabla \tilde{u}_2) \| \leq L(\Omega',\Omega',\Omega') \rightarrow 0.
\]

By the fact that $\nabla u_1$ is independent of $y$ of Proposition 2 we can get the proof of Theorem 2.

References


