一类对数形式退缩抛物型方程组解的整体存在性与爆破

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摘 要 考虑了一类由对数函数产生的退缩抛物型方程组的初边值问题,利用逼近的方法得到了解的局部存在性. 通过特征函数构造的上解得到了解整体存在的条件,利用特征函数得到了解在有限时刻爆破的条件及爆破时间的上限.

关键词 退缩抛物型方程组; 局部解; 整体解; 爆破
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Global Existence and Blow-up for a Degenerate Parabolic System Come from Logarithmic Function

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Abstract In this paper, a class of initial-boundary value problem of degenerate parabolic system come from logarithmic function was considered. The local existence of solution was proved by approximation method. The condition of global existence of the solution was obtained by the upper solution of the characteristic function, and the blow-up condition and the upper bound of the blow-up time were obtained by using the characteristic function.

Keywords degenerate parabolic system; local solution; global solution; blow-up

本文考虑如下由对数函数产生的退缩抛物型方程组解的初边值问题:

$$\begin{cases} u_t = (1+u)[\ln(1+u)]^{\alpha}(\Delta u + f(v)), \\ \qquad x \in \Omega, t > 0, \\ v_t = (1+v)[\ln(1+v)]^{\beta}(\Delta v + g(u)), \\ \qquad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), x \in \Omega, \\ u(x,t) = 0, v(x,t) = 0, x \in \partial\Omega, t > 0. \end{cases} \tag{1}$$

其中 Ω 为 R^N($N \geqslant 1$ 为正整数) 中具有光滑边界 $\partial\Omega$ 的有界区域, $u_0(x)$, $v_0(x)$ 为 Ω 上非负连续可微函数, 在 $\partial\Omega$ 上, $u_0(x) = 0$, $v_0(x) = 0$, $f(s)$, $g(s)$ 在 $s \geqslant 0$ 时为非负连续可微函数, 常数 $\alpha, \beta > 0$.

对于抛物型方程与方程组在有界区域上的初边值问题解的存在性与爆破性的讨论已有很长时间了,早期的讨论是针对半线性方程与方程组进行,主要讨论整体解存在的条件以及当整体解不存在而在有限时刻发生爆破时对爆破时间上限的估计[1-4]. 随着讨论的深入,对爆破点集的分布和爆破速率的估计也开始进行并逐步精确[5-7]; 另一方面讨论的方程及方程组的复杂程度逐渐加大,从最初的半线性方程到后来的退缩方程[8-11],再到不同耦合形式的方程组[12-14],以及各种类型的退缩方程组[15,16]. 对于退缩性是由幂函数引起的方程及方程组的讨论已有不少成果,文[15] 中讨论的方程组是 $u_t = v^{\alpha}(u_{xx} + au)$, $v_t = u^{\alpha}(v_{xx} + bv)$, 文[16] 中讨论的方程组是 $u_t = u^p(\Delta u + av)$, $v_t = v^q(\Delta v + bu)$, 文[17] 对文[16] 中的方程组在无界区域上进行了讨论.

本文讨论的初边值问题(1) 中,方程的退缩性是由对数函数引起,当 $u = 0$ 或 $v = 0$ 时,方程出现退

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缩. 首先利用逼近的方法给出解的局部存在性，其次讨论解整体存在的条件，最后讨论解在有限时刻发生爆破的条件.

1 解的局部存在性

对于退缩抛物型方程组初边值问题解的局部存在性与唯一性，讨论的方法有几种，但基本相同，在此只简要介绍主要步骤，不作详细的证明. 首先，构造一个近似的非退缩抛物方程组的初边值问题. 对常数 $\varepsilon > 0$，考虑如下问题:

$$\begin{cases} u_{\varepsilon t} = A_\varepsilon(u_\varepsilon)(\Delta u_\varepsilon + f(v_\varepsilon)), x \in \Omega, t > 0, \\ v_{\varepsilon t} = B_\varepsilon(v_\varepsilon)(\Delta v_\varepsilon + g(u_\varepsilon)), x \in \Omega, t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon, v_\varepsilon(x,0) = v_0(x) + \varepsilon, \\ \qquad x \in \Omega, \\ u_\varepsilon(x,t) = \varepsilon, v_\varepsilon(x,t) = \varepsilon, x \in \partial\Omega, t > 0. \end{cases} \tag{2}$$

其中 $A(u_\varepsilon)$，$B(v_\varepsilon)$ 为处处满足 $A(u_\varepsilon) \geqslant (1 + \frac{\varepsilon}{2})[\ln(1 + \frac{\varepsilon}{2})]^\alpha$，$B(v_\varepsilon) \geqslant (1 + \frac{\varepsilon}{2})[\ln(1 + \frac{\varepsilon}{2})]^\beta$ 的光滑函数，且满足:

$$A_\varepsilon(u_\varepsilon) = \begin{cases} (1 + u_\varepsilon)[\ln(1 + u_\varepsilon)]^\alpha, u_\varepsilon \geqslant \varepsilon, \\ (1 + \frac{\varepsilon}{2})[\ln(1 + \frac{\varepsilon}{2})]^\alpha, u_\varepsilon < \frac{\varepsilon}{2}. \end{cases}$$

$$B_\varepsilon(v_\varepsilon) = \begin{cases} (1 + v_\varepsilon)[\ln(1 + v_\varepsilon)]^\beta, v_\varepsilon \geqslant \varepsilon, \\ (1 + \frac{\varepsilon}{2})[\ln(1 + \frac{\varepsilon}{2})]^\beta, v_\varepsilon < \frac{\varepsilon}{2}. \end{cases}$$

初边值问题(2)为非退缩的拟线性抛物型方程组，由经典的抛物型方程理论可知，其解是局部存在的，即存在 $\sigma > 0$，使问题(2)在 $\Omega \times [0,\sigma)$ 上至少存在一个正解，仍记为 $(u_\varepsilon, v_\varepsilon)$，再由解的延展知，存在最大值 T_ε，使问题(2)在 $\Omega \times [0, T_\varepsilon)$ 上至少存在一个正解，且由极大值原理知 $u_\varepsilon \geqslant \varepsilon$，$v_\varepsilon \geqslant \varepsilon$，于是 $(u_\varepsilon, v_\varepsilon)$ 是如下初边值问题的解:

$$\begin{cases} u_{\varepsilon t} = (1 + u_\varepsilon)[\ln(1 + u_\varepsilon)]^\alpha(\Delta u_\varepsilon + f(v_\varepsilon)), \\ \qquad x \in \Omega, t > 0, \\ v_{\varepsilon t} = (1 + v_\varepsilon)[\ln(1 + v_\varepsilon)]^\beta(\Delta v_\varepsilon + g(u_\varepsilon)), \\ \qquad x \in \Omega, t > 0, \\ u_\varepsilon(x,0) = u_0(x) + \varepsilon, v_\varepsilon(x,0) = v_0(x) + \varepsilon, \\ \qquad x \in \Omega, \\ u_\varepsilon(x,t) = \varepsilon, v_\varepsilon(x,t) = \varepsilon, x \in \partial\Omega, t > 0. \end{cases}$$

其次，由比较原理[18]，当 $\varepsilon_1 > \varepsilon_2$ 时，对应的解满足 $u_{\varepsilon 1} \geqslant u_{\varepsilon 2}$，$v_{\varepsilon 1} \geqslant v_{\varepsilon 2}$，即 $(u_\varepsilon, v_\varepsilon)$ 随 ε 的减少而减少，于是，存在 $T > 0$，当 $\varepsilon \to 0^+$ 时，$T_\varepsilon \to T$，$(u_\varepsilon, v_\varepsilon) \to (u, v)$，由此得到问题(1)解的存在性. 进一步通过经典的方法还可以得到解的唯一性[18]，即定理 1.

定理 1 存在时刻 T，使问题(1)在 $\Omega \times (0, T)$ 上存在唯一的局部解 (u, v)，其中 $0 < T \leqslant +\infty$. 如果 $T < +\infty$，则有 $\overline{\lim\limits_{t \to T^-}} \max\limits_{x \in \Omega} u(x,t) = \infty$，或 $\overline{\lim\limits_{t \to T^-}} \max\limits_{x \in \Omega} v(x,t) = \infty$，即解在有限时刻发生爆破.

2 解的整体存在性

定义 1 如果 $(\bar{u}(x,t), \bar{v}(x,t))$ 满足下面的不等式，就称为问题(1)的一个上解:

$$\begin{cases} \bar{u}_t - (1 + \bar{u})[\ln(1 + \bar{u})]^\alpha(\Delta\bar{u} + f(\bar{v})) \geqslant 0, \\ \qquad x \in \Omega, t > 0, \\ \bar{v}_t - (1 + \bar{v})[\ln(1 + \bar{v})]^\beta(\Delta\bar{v} + g(\bar{u})) \geqslant 0, \\ \qquad x \in \Omega, t > 0, \\ \bar{u}(x,0) \geqslant u_0(x), \bar{v}(x,0) \geqslant v_0(x), x \in \Omega, \\ \bar{u}(x,t) \geqslant 0, \bar{v}(x,t) \geqslant 0, x \in \partial\Omega, t > 0. \end{cases}$$

如果 $(\underline{u}(x,t), \underline{v}(x,t))$ 满足与上面不等式中不等号相反的不等式，就称为问题(1)的一个下解.

由比较原理[18]，容易得到下面的引理 1.

引理 1 设 $(\bar{u}, \bar{v})$ 为问题(1)的一个上解，$(\underline{u}, \underline{v})$ 是问题(1)的一个下解，而 (u, v) 为问题(1)的解，那么:

$\underline{u}(x,t) \leqslant u(x,t) \leqslant \bar{u}(x,t)$，$\underline{v}(x,t) \leqslant v(x,t) \leqslant \bar{v}(x,t)$，$x \in \Omega$，$t > 0$.

设 $\varphi(x)$ 是特征值问题:

$$\begin{cases} \Delta\varphi(x) + \lambda\varphi(x) = 0, x \in \Omega, \\ \varphi(x) = 0, x \in \partial\Omega. \end{cases} \tag{3}$$

的第一特征函数，λ_1 为对应的第一特征值，则由特征值理论知 $\lambda_1 > 0$，在 Ω 内 $\varphi(x) > 0$，在此，取 $\max\limits_{x \in \Omega} \varphi(x) = 1$.

定理 2 设存在正常数 a, b 和 k, l，且 $al \leqslant \lambda_1 k$，$bk \leqslant \lambda_1 l$，使函数 $f(s)$，$g(s)$ 满足: $f(s) \leqslant as$，$g(s) \leqslant bs$，$s > 0$，函数 $u_0(x)$，$v_0(x)$ 满足 $u_0(x) \leqslant k\varphi(x)$，$v_0(x) \leqslant l\varphi(x)$，$x \in \Omega$，则问题(1)存在整体解.

证明 令 $\bar{u}(x,t) = k\varphi(x)$，$\bar{v}(x,t) = l\varphi(x)$，则当 $t = 0$，$x \in \Omega$ 时有 $\bar{u}(x,0) = k\varphi(x) \geqslant u_0(x)$，$\bar{v}(x,0) = l\varphi(x) \geqslant v_0(x)$. 当 $x \in \partial\Omega$，$t > 0$ 时，有 $\bar{u}(x,t) \geqslant 0$，$\bar{v}(x,t) \geqslant 0$，而当 $x \in \Omega$，$t > 0$ 时有 $\bar{u}_t - (1 + \bar{u})[\ln(1 + \bar{u})]^\alpha(\Delta\bar{u} + f(\bar{v})) = -(1 + k\varphi)[\ln(1 + k\varphi)]^\alpha(k\Delta\varphi + f(l\varphi))$，利用(3)式，有:

$\bar{u}_t - (1 + \bar{u})[\ln(1 + \bar{u})]^\alpha(\Delta\bar{u} + f(\bar{v})) =$

$$(1+k\varphi)[\ln(1+k\varphi)]^{\alpha}(\lambda_1 k\varphi - f(l\varphi)) \geqslant$$
$$(1+k\varphi)[\ln(1+k\varphi)]^{\alpha}(\lambda_1 k\varphi - al\varphi) \geqslant 0,$$

同理,有: $\bar{v}_t - (1+\bar{v})[\ln(1+\bar{v})]^{\beta}(\Delta\bar{v}+g(\bar{u})) = -(1+l\varphi)[\ln(1+l\varphi)]^{\beta}(l\Delta\varphi + g(k\varphi))$,

利用(3)式,有:

$$\bar{v}_t - (1+\bar{v})[\ln(1+\bar{v})]^{\beta}(\Delta\bar{v}+g(\bar{u})) =$$
$$(1+l\varphi)[\ln(1+l\varphi)]^{\beta}(\lambda_1 l\varphi - g(k\varphi)) \geqslant$$
$$(1+l\varphi)[\ln(1+l\varphi)]^{\beta}(\lambda_1 l\varphi - bk\varphi) \geqslant 0.$$

于是问题(1)存在上解$(k\varphi(x), l\varphi(x))$,由引理1得$u(x,t) \leqslant k\varphi(x)$,$v(x,t) \leqslant l\varphi(x)$对任意$t>0$都成立,故$u(x,t)$,$v(x,t)$对任意$t>0$都有定义,证毕.

3　解的爆破性

引理2　设初值函数$u_0(x)$,$v_0(x)$满足:

$$u_0(x) \geqslant \varphi(x),\ v_0(x) \geqslant \varphi(x), \tag{4}$$

且存在常数$a>\lambda_1$,$b>\lambda_1$,使函数$f(s)$,$g(s)$满足:

$$f(s) \geqslant as,\ g(s) \geqslant bs,\ s>0, \tag{5}$$

则问题(1)的解满足$u(x,t) \geqslant \varphi(x)$,$v(x,t) \geqslant \varphi(x)$,$x \in \Omega$,$t>0$.

证明　令$w(x,t) = u(x,t) - \varphi(x)$,$z(x,t) = v(x,t) - \varphi(x)$,则$t=0$时,$w \geqslant 0$,$z \geqslant 0$,$x \in \partial\Omega$时,$w = z = 0$,而$x \in \Omega$,$t>0$时,$w_t = u_t = (1+u)[\ln(1+u)]^{\alpha}(\Delta u + f(v)) = (1+u)[\ln(1+u)]^{\alpha}(\Delta w + \Delta\varphi + f(v))$,

利用(3)、(5)式,有:

$w_t = (1+u)[\ln(1+u)]^{\alpha}\Delta w + (1+u)[\ln(1+u)]^{\alpha}(f(v) - \lambda_1\varphi) \geqslant (1+u)[\ln(1+u)]^{\alpha}\Delta w + (1+u)[\ln(1+u)]^{\alpha}(av - \lambda_1\varphi) = (1+u)[\ln(1+u)]^{\alpha}\Delta w + (1+u)[\ln(1+u)]^{\alpha}(az + a\varphi - \lambda_1\varphi)$,

即有: $w_t \geqslant (1+u)[\ln(1+u)]^{\alpha}\Delta w +$

$$(1+u)[\ln(1+u)]^{\alpha}az. \tag{6}$$

$z_t = v_t = (1+v)[\ln(1+v)]^{\beta}(\Delta v + g(u)) = (1+v)[\ln(1+v)]^{\beta}(\Delta z + \Delta\varphi + g(u))$,

利用(3)、(5)式,有:

$z_t = (1+v)[\ln(1+v)]^{\beta}\Delta z + (1+v)[\ln(1+v)]^{\beta}(g(u) - \lambda_1\varphi) \geqslant (1+v)[\ln(1+v)]^{\beta}\Delta z + (1+v)[\ln(1+v)]^{\beta}(bu - \lambda_1\varphi) = (1+v)[\ln(1+v)]^{\beta}\Delta z + (1+v)[\ln(1+v)]^{\beta}(bw + b\varphi - \lambda_1\varphi)$,

即有: $z_t \geqslant (1+v)[\ln(1+v)]^{\beta}\Delta z +$

$$(1+v)[\ln(1+v)]^{\beta}bw. \tag{7}$$

由(6)、(7)式及抛物方程组的极大值原理知$w \geqslant 0$,$z \geqslant 0$,证毕.

定理3　设$\alpha,\beta>1$,函数$u_0(x)$,$v_0(x)$与$f(s)$,$g(s)$满足(4)和(5)式,则存在有限时刻T,使问题(1)的解在时刻T爆破,记

$$T_0 = \frac{\int_{\Omega}([\ln(1+u_0(x))]^{1-\alpha} + [\ln(1+v_0(x))]^{1-\beta})\varphi(x)\,\mathrm{d}x}{((b-\lambda_1)(\beta-1) + (a-\lambda_1)(\alpha-1))\int_{\Omega}\varphi^2(x)\,\mathrm{d}x},$$

则爆破时刻满足$T \leqslant T_0$.

证明　令$F(t) = \int_{\Omega}([\ln(1+u(x,t))]^{1-\alpha} + [\ln(1+v(x,t))]^{1-\beta})\varphi(x)\,\mathrm{d}x$,则有:

$$F'(t) = (1-\alpha)\int_{\Omega}(1+u(x,t))^{-1}[\ln(1+u(x,t))]^{-\alpha}u_t(x,t)\varphi(x)\,\mathrm{d}x + (1-\beta)\int_{\Omega}(1+v(x,t))^{-1}[\ln(1+v(x,t))]^{-\beta}v_t(x,t)\varphi(x)\,\mathrm{d}x =$$
$$-(\alpha-1)\int_{\Omega}(\Delta u + f(v))\varphi(x)\,\mathrm{d}x - (\beta-1)\int_{\Omega}(\Delta v + g(u))\varphi(x)\,\mathrm{d}x,$$

由格林公式及(3)式,有:

$$F'(t) = -(\alpha-1)\int_{\Omega}(u\Delta\varphi(x) + f(v)\varphi(x))\,\mathrm{d}x - (\beta-1)\int_{\Omega}(v\Delta\varphi + g(u)\varphi(x))\,\mathrm{d}x =$$
$$(\alpha-1)\int_{\Omega}\lambda_1 u(x,t)\varphi(x)\,\mathrm{d}x - (\alpha-1)\int_{\Omega}f(v)\varphi(x)\,\mathrm{d}x + (\beta-1)\int_{\Omega}\lambda_1 v(x,t)\varphi(x)\,\mathrm{d}x - (\beta-1)\int_{\Omega}g(u)\varphi(x)\,\mathrm{d}x,$$

由(4)、(5)式得:

$$F'(t) \leqslant (\alpha-1)\int_{\Omega}\lambda_1 u(x,t)\varphi(x)\,\mathrm{d}x - (\alpha-1)\int_{\Omega}av(x,t)\varphi(x)\,\mathrm{d}x + (\beta-1)\int_{\Omega}\lambda_1 v(x,t)\varphi(x)\,\mathrm{d}x - (\beta-1)\int_{\Omega}bu(x,t)\varphi(x)\,\mathrm{d}x = -(b(\beta-1) - \lambda_1(\alpha-1))\int_{\Omega}u(x,t)\varphi(x)\,\mathrm{d}x - (a(\alpha-1) - \lambda_1(\beta-1))\int_{\Omega}v(x,t)\varphi(x)\,\mathrm{d}x.$$

由引理2得:

$$F'(t) \leqslant -(b(\beta-1) - \lambda_1(\alpha-1))\int_{\Omega}\varphi^2(x)\,\mathrm{d}x - (a(\alpha-1) - \lambda_1(\beta-1))\int_{\Omega}\varphi^2(x)\,\mathrm{d}x.$$

最后一式右边为一常数,记为$-c$,其中$c>0$,于是有$F'(t) \leqslant -c$,两边积分得$F(t) - F(0) \leqslant -ct$,于是$ct \leqslant F(0)$,即$F(t)$的定义区间有限,故问题(1)的解$u(x,t)$,$v(x,t)$必在有限时刻发生爆破,且爆破时刻$T$满足$T \leqslant \frac{F(0)}{c} = T_0$,证毕.

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