可穿透腔体外有裂缝的正散射问题

彭超权,陶 婷,蔡明建
(中南民族大学 数学与统计学学院,武汉 430074)

摘 要 研究了时间调和的点源入射平面波通过腔体和裂缝的正散射问题,认为散射体是由一个可穿透腔体和一个外部不可穿透的裂缝构成,该问题归结为对具有一定边界条件的 Helmholtz 方程的求解. 通过边界积分方程的方法,利用位势理论和 Fredholm 定理,证明了该问题解的存在唯一性.

关键词 边界积分方程的方法; Helmholtz 方程; Fredholm 定理; 腔体; 裂缝
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The Direct Scattering Problem for a Penetrable Cavity and a Crack

Peng Chaoquan, Tao Ting, Cai Mingjian
(College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China)

Abstract In this paper, by using a time harmonic point source as the incident wave, we consider the scattering problem of a mixed scatterer composed of a penetrable cavity and an external impenetrable crack. The problem comes down to solving the Helmholtz equation with certain boundary conditions. By using the boundary integral equation method, based on the potential theory and Fredholm theorem, we prove that the scattering problem has a unique solution.

Keywords boundary integral equation method; Helmholtz equation; Fredholm theorem; cavity; crack

1 主要问题及结论

本文主要考虑关于时间调和的点源入射平面波通过腔体和裂缝的正散射问题. 散射体是由一个可穿透腔体和一个外部不可穿透的裂缝构成. 用有界且具有光滑边界的区域 D_1 来表示腔体,腔体的边界记作 S_1,用开弧 Γ 表示裂缝. 开弧 Γ 能延拓成光滑封闭单联通曲线 S_2,设其包含的区域为 D_2,满足 $D_1 \cap D_2 = \varnothing$. 记 $D: = R^2\backslash\overline{D}_1$ 为无界的光滑区域. 利用放置于腔体内部的点源 u^i 作为入射波,那么在 D_1 内形成散射场 u^s,在 D_1 外形成透射场 v. 波场在 S_1, S_2 满足不同的边界条件. 波场可用如下 Helmholtz 方程和一定的边界条件来描述:

$$\begin{cases}\Delta u + k_1^2 u = 0, \text{in } D_1, \\ \Delta v + k_2^2 v = 0, \text{in } D\backslash\overline{\Gamma}, \\ v_+ - u_- = 0, \text{on } S_1, \\ \dfrac{\partial v_+}{\partial \nu} - \dfrac{\partial u_-}{\partial \nu} = 0, \text{on } S_1, \\ v_+ = v_- = 0, \text{on } \Gamma.\end{cases} \tag{1}$$

正数 k_1, k_2 分别是区域 D_1 和 $D\backslash\overline{\Gamma}$ 内的波数,ν 为 S_1 外单位的法向量. 总场 $u = u^i + u^s$,$u^i = \dfrac{i}{4}H_0^1(k|x - y|)$,$v$ 满足 Sommerfeld 衰减条件:

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial v}{\partial r} - ik_2 v\right) = 0, \tag{2}$$

上式对 $\hat{x} = x\backslash|x|$ 一致成立($r = |x|$).

1995 年,Kress 研究了裂缝散射的正反散射问题,他利用边界积分方程的方法,得到了解的存在唯

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作者简介 彭超权(1979-),男,副教授,博士,研究方向:偏微分方程,E-mail:pcq1979@163.com
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一性[1]. 1997年,Monch考虑了具有Neumann边界条件的裂缝散射问题,之后在2000年,Kirsch和Ritter通过远场信息对裂缝进行了重构. 2003年,Cakoni和Colton考虑了裂缝两边具有不同边界条件的裂缝散射问题[2,3]. 2009年,Krutitskii研究了平面上一类裂缝的Helmholtz方程的边界值问题. 更多关于裂缝散射的问题大家可以参见文献[4-8]. 通常腔体的散射问题可用有界障碍物内的有界区域的Helmholtz方程来刻画,腔体外有障碍物的正散射问题在文献[9]有详细的讨论,更多关于腔体的正反散射问题的研究可参见文献[10-14],由于反散射问题的研究需要以正散射问题作为坚实的理论基础,本文将借助Kress的边界积分方程的方法,在恰当的Sobolev空间中考虑腔体外有裂缝的正散射问题,将问题(1)、(2)转化为一个边界积分系统,并证明边界积分算子Fredholm性和单射性,从而得到解的存在唯一性及连续依赖性结论.

为了叙述的方便,先介绍几个Sobolev空间和对应的迹空间. Sobolev空间有$H^1(D)$,$H^1_{loc}(R^2\backslash\overline{D})$,其迹空间就是$H^{1/2}(\partial D)$. 假设$\Omega$是有界区域,$\Sigma$是$\partial\Omega$的开子集,另外定义:

$$L^2(\Sigma)=\{u|_\Sigma : u\in L^2(\partial\Omega)\},$$
$$H^{1/2}(\Sigma)=\{u|_\Sigma : u\in H^{1/2}(\partial\Omega)\},$$
$$\widetilde{H}^{1/2}(\Sigma)=\{u\in H^{1/2}(\partial\Omega):\operatorname{supp}u\in\bar{\Sigma}\},$$

$H^{-1/2}(\Sigma)=(\widetilde{H}^{1/2}(\Sigma))'$表示$\widetilde{H}^{1/2}(\Sigma)$的对偶空间,根据文献[15]有:

$$\widetilde{H}^{1/2}(\Sigma)\subset H^{1/2}(\Sigma)\subset L^2(\Sigma)\subset\widetilde{H}^{-1/2}(\Sigma)\subset H^{-1/2}(\Sigma).$$

考虑一个更为一般的问题,对于给定的$h\in L^2(D\backslash\overline{\Gamma})$,$p\in H^{1/2}(S_1)$,$q\in H^{-1/2}(S_1)$,$f\in H^{1/2}(\Gamma)$,寻求$u\in H^1_{loc}(D\backslash\overline{\Gamma})\cap H^1(D_1)$,满足:

$$\begin{cases}\Delta u+k_1^2u=0,\text{in } D_1,\\ \Delta u+k_2^2u=0,\text{in } D\backslash\overline{\Gamma},\\ u_+-u_-=p,\text{on } S_1,\\ \dfrac{\partial u_+}{\partial\nu}-\dfrac{\partial u_-}{\partial\nu}=q,\text{on } S_1,\\ u_+=u_-=f,\text{on } \Gamma.\end{cases}\qquad(3)$$

并且u满足Sommerfeld衰减条件(2).

本文的主要结果是定理1和定理2.

定理1 问题(2)、(3)至多有一个解.

定理2 假设$p\in H^{1/2}(S_1)$,$q\in H^{-1/2}(S_1)$,$f\in H^{1/2}(\Gamma)$,则问题(2)、(3)存在唯一的解$u\in H^1_{loc}(D\backslash\overline{\Gamma})\cap H^1(D_1)$,且满足下列估计:

$$\|u\|_{H^1(D_1)}+\|u\|_{H^1_{loc}(D\backslash\overline{\Gamma})}\leqslant C(\|p\|_{H^{1/2}(S_1)}+\|q\|_{H^{-1/2}(S_1)}+\|f\|_{H^{1/2}(\Gamma)}).\qquad(4)$$

2 定理1的证明

定理1的证明 事实上,如果$h=p=q=f=0$,我们若能证明$u\equiv0$,则结论成立.

首先,作一个以原点为中心、R为半径的充分大的球B_R,使其包含区域$\overline{D}_1$和$\overline{D}_2$,并用∂B_R表示该球的边界. 设$u\in H^1(D_1)\cap H^1(B_R\backslash\{\overline{D}_1\cup\overline{\Gamma}\})$为齐次问题(3)的解,显然在边界$\partial D_2\backslash\overline{\Gamma}$上满足:

$$\begin{cases}u_+=u_-,\\ \dfrac{\partial u_+}{\partial\nu}=\dfrac{\partial u_-}{\partial\nu}.\end{cases}\qquad(5)$$

其中"±"分别表示从区域D_2的外部和内部逼近边界∂D_2的极限. 对u和$\bar{u}$分别在区域$B_R\backslash(\overline{D}_1\cup\overline{D}_2)$,$D_1$,$D_2$应用Green公式,可以得到:

$$\int_{B_R\backslash(\overline{D}_1\cup\overline{D}_2)}(u\Delta\bar{u}+\nabla u\cdot\nabla\bar{u})\,dx=\int_{\partial B_R}\frac{\partial\bar{u}}{\partial\nu}ds-\int_{S_1}u_+\frac{\partial\bar{u}_+}{\partial\nu}ds-\int_{S_2}u_+\frac{\partial\bar{u}_+}{\partial\nu}ds,$$
$$\int_{D_1}(u\Delta\bar{u}+\nabla u\cdot\nabla\bar{u})\,dx=\int_{S_1}u_-\frac{\partial\bar{u}_-}{\partial\nu}ds,$$
$$\int_{D_2}(u\Delta\bar{u}+\nabla u\cdot\nabla\bar{u})\,dx=\int_{S_2}u_-\frac{\partial\bar{u}_-}{\partial\nu}ds.$$

利用边界条件,把上式相加可得:

$$\int_{B_R\backslash(\overline{D}_1\cup\overline{D}_2)}(|\nabla u|^2-k_2^2u^2)\,dx+\int_{D_1}(|\nabla u|^2-k_1^2u^2)\,dx+\int_{D_2}(|\nabla u|^2-k_2^2u^2)\,dx=\int_{\partial B_R}u\frac{\partial\bar{u}}{\partial\nu}ds,$$

从而$\operatorname{Im}(\int_{\partial B_R}u\frac{\partial\bar{u}}{\partial\nu}ds)\geqslant0$,由Rellich引理可知$u=0$,$x\in D\backslash\overline{\Gamma}$. 因此有$\frac{\partial u_+}{\partial\nu}=0$和$u_+=0$,$x\in S_1$. 由边界条件(5)可知$\frac{\partial u_-}{\partial\nu}=0$和$u_-=0$,$x\in S_1$. 由Holmgren唯一性定理$u=0$,$x\in D_1$.

3 定理2的证明

利用边界积分方程的方法来证明问题(2)、(3)解的存在性. 由Green表示公式有:

$u(x)=$

$$\begin{cases}\int_{S_1}\left[\frac{\partial u}{\partial \upsilon}\Phi_1(x,y)-u\frac{\partial \Phi_1(x,y)}{\partial \upsilon}\right]\mathrm{d}S_y, x\in D_1,\\ -\int_{S_1}\left[\frac{\partial u}{\partial \upsilon}\Phi_2(x,y)-u\frac{\partial \Phi_2(x,y)}{\partial \upsilon}\right]\mathrm{d}S_y-\\ \int_{S_2}\left[\frac{\partial u}{\partial \upsilon}\Phi_2(x,y)-u\frac{\partial \Phi_2(x,y)}{\partial \upsilon}\right]\mathrm{d}S_y,\\ \quad x\in D\backslash\overline{D}_2,\\ \int_{S_2}\left[\frac{\partial u}{\partial \upsilon}\Phi_2(x,y)-u\frac{\partial \Phi_2(x,y)}{\partial \upsilon}\right]\mathrm{d}S_y, x\in D_2.\end{cases} \tag{6}$$

利用单双层位势穿过边界 S_1 的跳跃关系(参见文献[16]的第三章),在区域 D_1 上,考虑当 u 从 D_1 逼近边界 S_1 时,有:

$$u_-=S_{1,s_1s_1}\frac{\partial u_-}{\partial \upsilon}-K_{1,s_1s_1}u_-,\frac{\partial u_-}{\partial \upsilon}=K'_{1,s_1s_1}\frac{\partial u_-}{\partial \upsilon}-T_{1,s_1s_1}u_-, x\in S_1, \tag{7}$$

在区域 D 上,考虑 u 从 D 逼近边界 S_1,有:

$$u_+=-\left(S_{2,s_1s_1}\frac{\partial u_+}{\partial \upsilon}-K_{2,s_1s_1}u_++S_{2,s_2s_1}\frac{\partial u_+}{\partial \upsilon}-K_{2,s_2s_1}u_+\right), x\in S_1, \tag{8}$$

$$\frac{\partial u_+}{\partial \upsilon}=-\left(K'_{2,s_1s_1}\frac{\partial u_+}{\partial \upsilon}-T_{2,s_1s_1}u_++K'_{2,s_2s_1}\frac{\partial u_+}{\partial \upsilon}-T_{2,s_2s_1}u_+\right), x\in S_1, \tag{9}$$

在区域 D 上,考虑 u 从 D 逼近边界 S_2,有:

$$u_+=-\left(S_{2,s_1s_2}\frac{\partial u_+}{\partial \upsilon}-K_{2,s_1s_2}u_++S_{2,s_2s_2}\frac{\partial u_+}{\partial \upsilon}-K_{2,s_2s_2}u_+\right), x\in S_2, \tag{10}$$

$$\frac{\partial u_+}{\partial \upsilon}=-\left(K'_{2,s_1s_2}\frac{\partial u_+}{\partial \upsilon}-T_{2,s_1s_2}u_++K'_{2,s_2s_2}\frac{\partial u_+}{\partial \upsilon}-T_{2,s_2s_2}u_+\right), x\in S_2, \tag{11}$$

在区域 D_2 上,考虑当 u 从 D_2 逼近边界 S_2 时,有:

$$u_-=S_{2,s_2s_2}\frac{\partial u_-}{\partial \upsilon}-K_{2,s_2s_2}u_-,\frac{\partial u_-}{\partial \upsilon}=K'_{1,s_2s_2}\frac{\partial u_-}{\partial \upsilon}-T_{1,s_2s_2}u_-, x\in S_2, \tag{12}$$

其中:

$$S_{i,s_js_l}\varphi(x)=2\int_{S_j}\varphi(y)\Phi_i(x,y)\mathrm{d}S_y,$$

$$K_{i,s_js_l}\varphi(x)=2\int_{S_j}\varphi(y)\frac{\partial \Phi_i(x,y)}{\partial \upsilon_y}\mathrm{d}S_y, x\in S_l,$$

$$K'_{i,s_js_l}\varphi(x)=2\int_{S_j}\varphi(y)\frac{\partial \Phi_i(x,y)}{\partial \upsilon_x}\mathrm{d}S_y,$$

$$T_{i,s_js_l}\varphi(x)=2\frac{\partial}{\partial \upsilon_x}\int_{S_j}\varphi(y)\frac{\partial \Phi_i(x,y)}{\partial \upsilon_y}\mathrm{d}S_y, x\in S_l,$$

现在再来建立边界积分系统.

在边界 S_1 上定义:

$$\left.\frac{\partial u_-}{\partial \upsilon}\right|_{S_1}=a, u_-|_{S_1}=b.$$

由 S_1 的边界条件,有 $\frac{\partial u_+}{\partial \upsilon}=a+q, u_+=b+p$. 在裂缝 Γ 上,定义

$$[u]|_\Gamma=(u_+-u_-)|_\Gamma=c=0,$$

$$\left.\left[\frac{\partial u}{\partial \upsilon}\right]\right|_\Gamma=\left.\left(\frac{\partial u_+}{\partial \upsilon}-\frac{\partial u_-}{\partial \upsilon}\right)\right|_\Gamma=d.$$

把 c, d 延拓到整个边界 S_2:

$$\tilde{c}=\begin{cases}0, \text{on } S_2\backslash\Gamma,\\ c, \text{on } \Gamma,\end{cases}\quad \tilde{d}=\begin{cases}0, \text{on } S_2\backslash\Gamma,\\ d, \text{on } \Gamma.\end{cases}$$

参见文献[2],有 $\tilde{c}\in\widetilde{H}^{1/2}(\Gamma)$, $\tilde{d}\in\widetilde{H}^{-1/2}(\Gamma)$.

由(7)、(10)式以及 S_1 上的边界条件,我们有:

$$(u_+-u_-)|_{S_1}=-(S_{1,s_1s_1}+S_{2,s_1s_1})a+(K_{1,s_1s_1}+K_{2,s_1s_1})b-S_{2,s_1s_1}q+K_{2,s_1s_1}p-S_{2,s_2s_1}\tilde{d}+I_1, \tag{13}$$

其中 $I_1=-S_{2,s_2s_1}\frac{\partial u_-}{\partial \upsilon}+K_{2,s_2s_1}u_-$.

引理 1 I_1 的值为 0,也就是 $-S_{2,s_2s_1}\frac{\partial u_-}{\partial \upsilon}+K_{2,s_2s_1}u_-=0$.

证明 参考文献[3],利用 Green 表示公式即证.

定义 $r_1=+S_{2,s_1s_1}q-K_{2,s_1s_1}p+p$,(13)式即可写成:

$$-(S_{1,s_1s_1}+S_{2,s_1s_1})a+(K_{1,s_1s_1}+K_{2,s_1s_1})b-S_{2,\Gamma s_1}\tilde{d}=r_1. \tag{14}$$

由(7)、(9)式以及 S_1 上的边界条件有:

$$\left.\left(\frac{\partial u_+}{\partial \upsilon}-\frac{\partial u_-}{\partial \upsilon}\right)\right|_{S_1}=-(K'_{1,s_1s_1}+K'_{2,s_1s_1})a+(T_{1,s_1s_1}+T_{2,s_1s_1})b-K'_{2,s_1s_1}q+T_{2,s_1s_1}p-K'_{2,s_2s_1}\tilde{d}+I_2, \tag{15}$$

其中 $I_2=-K'_{2,s_2s_1}\frac{\partial u_-}{\partial \upsilon}+T_{2,s_2s_1}u_-$.

引理 2 I_2 的值为 0,也就是 $-K'_{2,s_2s_1}\frac{\partial u_-}{\partial \upsilon}+T_{2,s_2s_1}u_-=0$.

定义 $r_2=K'_{2,s_1s_1}q-T_{2,s_1s_1}p+q$,即可写成:

$$-(K'_{1,s_1s_1}+K'_{2,s_1s_1})a+(T_{1,s_1s_1}+T_{2,s_1s_1})b-K'_{2,\Gamma s_1}\tilde{d}=r_2. \tag{16}$$

由(9)、(11)式以及 S_2 上的边界条件,有:

$$u_++u_-=-S_{2,s_2s_2}\tilde{d}+K_{2,s_1s_2}p-S_{2,s_1s_2}q+I_3. \tag{17}$$

引理 3 I_3 的值为 0,也就是 $-S_{2,s_1s_2}\frac{\partial u_-}{\partial \upsilon}+K_{2,s_1s_2}u_-=0$. 定义 $r_3=-K_{2,s_1\Gamma}p+S_{2s_1\Gamma}q+2f+2\int_{D\backslash S_2}h\Phi_2(x,y)\mathrm{d}y$,即:

$$-S_{2,\Gamma\Gamma}\tilde{d}=r_3. \tag{18}$$

为了表述清楚,我们令:

$$A=\begin{pmatrix}(S_{1,s_1s_1}+S_{2,s_1s_1}) & -(K_{1,s_1s_1}+K_{2,s_1s_1}) & S_{2,\Gamma s_1}\\ (K'_{1,s_1s_1}+K'_{2,s_1s_1}) & -(T_{1,s_1s_1}+T_{2,s_1s_1}) & K'_{2,\Gamma s_1}\\ 0 & 0 & S_{2,\Gamma\Gamma}\end{pmatrix},$$

$$\vec{R}=\begin{pmatrix}-r_1\\-r_2\\-r_3\end{pmatrix},\chi=(a,b,\tilde{d})^{\mathrm{T}},$$

由(14)、(16)、(18)式,有边界积分系统:

$$A\chi=\vec{R}. \quad (19)$$

参见文献[3]的第七章,限制算子 $S_{2,\Gamma\Gamma}, K_{2,\Gamma\Gamma}, K'_{2,\Gamma\Gamma}, T_{2,\Gamma\Gamma}$ 有如下的映射性:

$$S_{2,\Gamma\Gamma}:\tilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma),$$
$$K_{2,\Gamma\Gamma}:\tilde{H}^{1/2}(\Gamma)\to H^{1/2}(\Gamma),$$
$$K'_{2,\Gamma\Gamma}:\tilde{H}^{-1/2}(\Gamma)\to H^{-1/2}(\Gamma),$$
$$T_{2,\Gamma\Gamma}:\tilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma).$$

其他的算子在积分系统(19)有连续的积分核. 故 A 从 H 连续映射到 H^*.

引理 4 算子 $A:H\mapsto H^*$ 是具有零指标的 Fredholm 算子,且具有平凡核.

证明 分为两步:第一步,证明算子 $A:H\mapsto H^*$ 是具有零指标的 Fredholm 算子. 第二步,证明 $\mathrm{Kern}\,A=\{0\}$.

第一步. 参考文献[5],我们知道存在相应的紧算子:

$L_{i,S}:H^{-1/2}(S_1)\to H^{1/2}(S_1), L_{i,T}:H^{1/2}(S_1)\to H^{-1/2}(S_1)$,
$J_{2,S}:H^{-1/2}(S_2)\to H^{1/2}(S_2), J_{2,T}:H^{1/2}(S_2)\to H^{-1/2}(S_2)$,
其中 $i=1,2$,使得:

$$\mathrm{Re}(\langle(S_{i,s_1s_1}+L_{i,S})\psi,\bar{\psi}\rangle)\geqslant C\|\psi\|^2_{H^{-1/2}(S_1)},\quad \psi\in H^{-1/2}(S_1), \quad (20)$$

$$\mathrm{Re}(\langle-(T_{i,s_1s_1}+L_{i,T})\psi,\bar{\psi}\rangle)\geqslant C\|\psi\|^2_{H^{1/2}(S_1)},\quad \psi\in H^{1/2}(S_1), \quad (21)$$

$$\mathrm{Re}(\langle(S_{2,s_2s_2}+J_{2,S})\psi,\bar{\psi}\rangle)\geqslant C\|\psi\|^2_{H^{-1/2}(S_2)},\quad \psi\in H^{-1/2}(S_2), \quad (22)$$

$$\mathrm{Re}(\langle-(T_{2,s_2s_2}+J_{2,T})\psi,\bar{\psi}\rangle)\geqslant C\|\psi\|^2_{H^{1/2}(S_2)},\quad \psi\in H^{1/2}(S_2), \quad (23)$$

其中 $\langle,\rangle$ 表示 $H^{-1/2}(S_1)$ 和 $H^{1/2}(S_1)$ 或者 $H^{-1/2}(S_2)$ 和 $H^{1/2}(S_2)$ 之间的对偶积. 我们定义:

$$S^*_{i,s_1s_1}=S_{i,s_1s_1}+L_{i,S}, i=1,2,$$
$$T^*_{i,s_1s_1}=-(T_{i,s_1s_1}+L_{i,T}), i=1,2,$$
$$S^*_{2,s_2s_2}=S_{2,s_2s_2}+J_{2,S},$$
$$T^*_{2,s_2s_2}=-(T_{2,s_2s_2}+J_{2,T}),$$

其中 $\Psi_0(x,y)=-\dfrac{1}{2\pi}\ln|x-y|, i,j=1,2$. 并且 $\tilde{K}_{i,s_js_j}=K_{i,s_js_j}-K^0_{i,s_js_j}$ 和 $\tilde{K}'_{i,s_js_j}=K'_{i,s_js_j}-K^{0'}_{i,s_js_j}$ 都是紧的且具有连续核(参见文献[3]第八章),可得 $K^0_{i,s_js_j}, K^{0'}_{i,s_js_j}$ 是一对共轭算子,

定义:$\tilde{A}=$

$$\begin{pmatrix}(S_{1,s_1s_1}+S_{2,s_1s_1}) & -(K_{1,s_1s_1}+K_{2,s_1s_1}) & -S_{2,s_2s_1}\\ (K'_{1,s_1s_1}+K'_{2,s_1s_1}) & -(T_{1,s_1s_1}+T_{2,s_1s_1}) & K'_{2,s_2s_1}\\ 0 & 0 & S_{2,s_2s_2}\end{pmatrix}$$

$\tilde{A}$ 从 $\tilde{H}$ 连续映射到 $\tilde{H}^*$,其中 $\tilde{H}=H^{-1/2}(S_1)\times H^{1/2}(S_1)\times H^{-1/2}(S_2)$,

将 $\tilde{A}$ 分解成两部分,即 $\tilde{A}\tilde{\chi}=\tilde{A}_0\tilde{\chi}+\tilde{A}_c\tilde{\chi}$,

$$\tilde{A}_0\tilde{\chi}=\begin{pmatrix}(S^*_{1,s_1s_1}+S^*_{2,s_1s_1})a-(K^0_{1,s_1s_1}+K^0_{2,s_1s_1})b\\ (K^{0'}_{1,s_1s_1}+K^{0'}_{2,s_1s_1})a+(T^*_{1,s_1s_1}+T^*_{2,s_1s_1})b\\ S^*_{2,s_2s_2}\tilde{d}\end{pmatrix},$$

$$\tilde{A}_c\tilde{\chi}=\begin{pmatrix}-(L_{1,S}+L_{2,S})a-(\tilde{K}_{1,s_1s_1}+\tilde{K}_{2,s_1s_1})b-S_{2,s_2s_1}\tilde{d}\\ (\tilde{K}'_{1,s_1s_1}+\tilde{K}'_{2,s_1s_1})a+(L_{1,T}+L_{2,T})b+K'_{2,s_2s_1}\tilde{d}\\ -J_{2,S}\tilde{d}\end{pmatrix},$$

其中 $\tilde{\chi}=(a,b,\tilde{d})^{\mathrm{T}}$ 是 $\chi=(a,b,d)^{\mathrm{T}}$ 的零拓展. 在这个分解中,$\tilde{A}_c:\tilde{H}\to\tilde{H}^*$ 是紧的,$\tilde{A}_0:\tilde{H}\to\tilde{H}^*$ 是半线性的,

$$\langle\tilde{A}_0\tilde{\chi},\tilde{\chi}\rangle_{\tilde{H},\tilde{H}^*}=((S^*_{1,s_1s_1}+S^*_{2,s_1s_1})a,a)-((K^0_{1,s_1s_1}+K^0_{2,s_1s_1})b,a)+((K^{0'}_{1,s_1s_1}+K^{0'}_{2,s_1s_1})a,b)+((T^*_{1,s_1s_1}+T^*_{2,s_1s_1})b,b)+(S^*_{2,s_2s_2}\tilde{d},\tilde{d}), \quad (24)$$

其中 (u,v) 表示在 $L^2(S_1)$ 或 $L^2(S_2)$ 的定义为 $\int_{S_1}u\bar{v}\mathrm{d}s$ 或 $\int_{S_2}u\bar{v}\mathrm{d}s$ 数量积,由(20)~(24)式得:

$$\mathrm{Re}[(S^*_{1,s_1s_1}a,a)+(S^*_{2,s_1s_1}a,a)+(T^*_{1,s_1s_1}b,b)+(T^*_{2,s_1s_1}b,b)+(S^*_{2,s_2s_2}\tilde{d},\tilde{d})]\geqslant C\|\tilde{\chi}\|^2_{\tilde{H}}. \quad (25)$$

由于 $K^0_{0,s_1s_1}, K^{0'}_{0,s_1s_1}$ 和 $K^0_{1,s_1s_1}, K^{0'}_{1,s_1s_1}$ 互为共轭的算子,我们有:

$$\mathrm{Re}[-((K^0_{1,s_1s_1}+K^0_{2,s_1s_1})b,a)+((K^{0'}_{1,s_1s_1}+K^{0'}_{2,s_1s_1})a,b)]=0. \quad (26)$$

由(25)、(26)式可知 $\tilde{A}_0$ 是强制的,$\mathrm{Re}(\langle(\tilde{A}-\tilde{A}_c)\tilde{\chi},\tilde{\chi}\rangle_{\tilde{H},\tilde{H}^*})\geqslant C\|\tilde{\chi}\|^2_{\tilde{H}}, \tilde{\chi}=\tilde{H}$. 由于 $\tilde{\chi}$ 是 χ 的零延拓,故 $|\langle(\tilde{A}_{0\Gamma}\chi,\tilde{\chi}\rangle|_{H^*,H}\geqslant C\|\chi\|^2_H, \chi\in H$,其中 $\tilde{A}_{0\Gamma}:H\mapsto H^*$ 表示 $\tilde{A}_0$ 限制在 Γ 上. $\tilde{A}_c$ 限制在 Γ 上的算子 $\tilde{A}_{c\Gamma}:H\mapsto H^*$ 显然是紧的. 由 Lax-Milgram 引理可知算子 $A:H\mapsto H^*$ 是具有零指标的 Fredholm 算子.

第二步,证明 $\mathrm{Kern}\,A=\{0\}$,我们仍用 $\chi=(a,$

$b, \tilde{d})^{\mathrm{T}}$ 表示 $A\chi = \vec{0}$ 的解，即：

$$\begin{cases} (S_{1,s_1s_1}, S_{2,s_1s_1})a - (K_{1,s_1s_1} + K_{2,s_1s_1})b + S_{2,\Gamma s_1}\tilde{d} = 0, \\ (K'_{1,s_1s_1} + K'_{2,s_1s_1})a - (T_{1,s_1s_1} + T_{2,s_1s_1})b + \\ \quad K'_{2,\Gamma s_1}\tilde{d} = 0, \\ S_{2,\Gamma\Gamma}\tilde{d} = 0. \end{cases} \tag{27}$$

构造位势：

$$v(x) = \int_{S_1} a\Phi_1(x,y)\,\mathrm{d}S_y - \int_{S_1} b\frac{\partial\Phi_1(x,y)}{\partial v(y)}\mathrm{d}S_y, x \in D, \tag{28}$$

$$w(x) = -\int_{S_1} a\Phi_2(x,y)\,\mathrm{d}S_y + \int_{S_1} b\frac{\partial\Phi_2(x,y)}{\partial v(y)}\mathrm{d}S_y - \int_{\Gamma}\tilde{d}\Phi_2(x,y)\,\mathrm{d}S_y, x \in D_1. \tag{29}$$

位势 $v(x)$ 和 $w(x)$ 在 D 和 D_1 满足不同的 Helmholtz 方程：

$$\begin{cases} \Delta v + k_1^2 v = 0, \text{in } D, \\ \Delta w + k_2^2 w = 0, \text{in } D_1, \end{cases}$$

由单双层位势在 S_1 的跳跃关系，有：

$$2v_+(x)|_{S_1} = S_{1,s_1s_1}a - K_{1,s_1s_1}b - b, 2w_-(x)|_{S_1} = -S_{1,s_1s_1}a + K_{1,s_1s_1}b - S_{1,\Gamma s_1}\tilde{d} - b, \tag{30}$$

$$\left.\frac{2\partial v_+(x)}{\partial v}\right|_{S_1} = K'_{1,s_1s_1}a - T_{1,s_1s_1}b - a, \left.\frac{2\partial w_-(x)}{\partial v}\right|_{S_1} = -K'_{2,s_1s_1}a + T_{2,s_1s_1}b - K'_{2,\Gamma s_1}\tilde{d} - a. \tag{31}$$

由(30)式和方程组(27)的第一个式子可知：

$$(v_+ - w_-)|_{S_1} = 0. \tag{32}$$

由(31)式和方程组(27)的第二个式子可知：

$$\left.\left(\frac{\partial v_+(x)}{\partial v} - \frac{\partial w_-(x)}{\partial v}\right)\right|_{S_1} = 0. \tag{33}$$

由(32)、(33)式，根据文献[1]及定理 1 可推导 $v(x) = 0, x \in D_1, w(x) = 0, x \in D\backslash\bar{\Gamma}$.

利用单双层位势的跳跃关系，有 $a = b = \tilde{d} = 0$. 故 $\chi = (0,0,0)^{\mathrm{T}}$，从而引理 4 得证.

定理 2 的证明　由引理 4 知：A 的逆算子 A^{-1}：$H^* \rightarrow H$ 存在且有界，再由位势函数(6)式即可得证.

参 考 文 献

[1] Kress R. Fréchet differentiability of the far field operator for scattering from a crack [J]. Journal of Inverse and Ill-posed Problems Series, 2009, 3(4): 305-313.

[2] Cakoni F, Colton D L. The linear sampling method for cracks [J]. Inverse Problems, 2003(19): 279-295.

[3] Cakoni F, Colton D L. Qualitative method in inverse scattering theory [M]. Berlin: Springer, 2006: 1-270.

[4] Kress R. Acoustic scattering, special theoretical tools [M]. London: Academic Press, 2001: 1-280.

[5] Yan G. Scattering problems by a partially coated crack [J]. Nonlinear Analysis, 2008(68): 932-939.

[6] Yan G, Ye J. Boundary integral methods for scattering problems with cracks buried in a piecewise homogeneous medium [J]. Mathematical Methods in the Applied Sciences, 2012(35): 84-96.

[7] Yan G, Yao M. Mathematical basis of scattering problems from penetrable obstacles and cracks [J]. Journal of Mathematical Physics, 2010(51): 123520.

[8] Yan G, Yao M. The method of boundary integral equations for a mixed scattering problem [J]. Journal of Differential Equations, 2009(246): 4618-4631.

[9] Liu X, Zhang B. Direct and inverse obstacle scattering problems in a piecewise homogeneous medium [J]. SIAM Journal on Applied Mathematics, 2010, 70(8): 3105-3120.

[10] Colton D, Kress R. Integral equation methods in scattering theory [M]. New York: Wiley, 1983: 1-271.

[11] Qin H H, Colton D. The inverse scattering problem for cavities [J]. Applied Numerical Mathematics, 2012, 62(2): 699-708.

[12] Liu X. The factorization method for cavities [J]. Inverse Problems, 2014, 30(1): 015006.

[13] Meng S, Haddar H, Cakoni F. The factorization method for cavity in an inhomogeneous medium [J]. Inverse Problems, 2014, 30(4): 045008.

[14] Hsiao G C, Wendland W L. Boundary integral equations [J]. Applied Mathematical Sciences, 2013, 76(4): 509-547.

[15] Mönch L. On the inverse acoustic scattering problem by an open arc: on the sound-hard case [J]. Inverse Problems, 1997, 13(5): 1379-1392.

[16] Colton D L, Kress R. Integral equation methods in scattering theory [M]. New York: Springer-Verlag, 1983: 1-271.