基于旋转变换的灰值形态算子

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摘 要 在二值旋转形态算子研究的基础上,研究了基于旋转变换的灰值形态算子的相关理论及方法,通过引入本影集合上的点积运算和灰值旋转变换等方法,将本影变换与表面算子应用于构建灰值旋转形态算子,提出并建立了基于空间旋转变换的灰值腐蚀、膨胀及开、闭算子,论证了一系列重要的性质,这些结论充实了基于旋转变换的形态学理论.

关键词 灰值旋转形态算子; 本影变换; 表面算子; 二值旋转形态算子
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Gray-Value Morphological Operators Based on Rotation Transform

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Abstract Based on the binary rotation morphological operators, the research on related theory and method of the gray-value morphological operator with rotation transform was proposed. By introducing the dot product operation of the umbra set and gray-value rotation transform for gray-value images, umbra transform and surface operator were applied to build gray-value rotation morphological transform, including erosion, dilation, opening and closing operator. The main structures and properties were considered, and the methods for solving them within the framework of rotation morphological operator were described, some important properties were obtained and they enriched the rotation morphology.

Keywords gray-value rotation morphological transform; umbra transform; surface operator; binary rotation morphological transform

Serra的形态学理论[1]建立在空间平移变换的基础之上,平移形态算子通过平移变换实现对目标对象几何结构的探测过程. 另一种常用的空间移动方式是旋转变换,在文献[2]中Heijmans提出了以旋转变换替代平移变换建立二值腐蚀、膨胀的思想. 文献[3]对此进行了详细的研究,建立了二值旋转形态算子的相关理论. 所有推证结果表明,在除去原点的空间中,二值旋转形态算子与二值平移形态算子具有相似的性质. 本文在文献[3]的基础上,进一步研究了基于旋转变换的灰值形态算子的相关理论及方法,通过引入本影集合上的点积运算和灰值形态旋转变换等方法,将本影变换与表面算子应用于构建灰值旋转形态算子,提出并建立了基于空间旋转变换的灰值腐蚀、膨胀及开、闭算子,论证了一系列重要的性质,充实了基于旋转变换的形态学理论. 研究结果表明:旋转形态算子与平移形态算子具有诸多类似的结构和性质[4-8],这一结果与Heijmans的T不变算子理论是一致的.

1 预备知识

设$E^n = R^n \setminus \{0\}$表示连续$n$维欧式空间剔除原点的集合. 灰值图像$y = f(x)$, $x \in E^n$其像素点x用极坐标(r,θ)描述,f定义域$D[f] \subset E^n$, $(x,f) \in E^{n+1}$,灰值图像f的全体构成的集合记为F. 灰值图像集合F中的序关系可按以下方式给出[2]:如果g和f的定义域满足条件$D[g] \subseteq D[f]$,且对于任意的$\forall x \in D[g]$有$g(x) \leqslant f(x)$(默认超出定义域范围

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的值为负无穷)，则称 g 位于 f 的下方，或称 g 小于 f，记为 $g \ll f$，由此可建立灰值图像的偏序集合 $(F, \ll)$.

在偏序集合 $(F, \ll)$ 中，可定义其上的极大和极小运算[2]:

$$(f \wedge g)(x) = \min\{f(x), g(x)\},\quad x \in D[f] \cap D[g],$$
$$(f \vee g)(x) = \max\{f(x), g(x)\},\quad x \in D[f] \cup D[g],$$

以及相对于旋转变换的反射运算: 图像 $h(x)$ 的反射 $\hat{h}(x)$ 定义为 $\hat{h}(x) = h(\hat{x})$，这里 $x \in D[\hat{h}]$ 等价于 $\hat{x} \in D[h]$，其中 $\hat{x}$ 是 x 的逆元[3].

2　旋转变换下阴影集和表面函数的性质

灰值图像函数 f 的本影 $U[f] = \{(x, y): x \in D[f], y \leqslant f(x)\}$[2,9]，若 $D[f] \subset E^n$，则 $U[f] \subseteq E^{n+1}$. 集合 A 的表面 $S[A] = \{(x, y) \in A: y \geqslant z, \forall (x, z) \in A\}$[2]，其表面函数具有形式 $S[A](x) = \max\{y \mid (x, y) \in A\}$. 显然，对于任意的灰值图像函数 f，其本影变换及表面函数具有如下关系[3]:

$$S(U[f])(x) = f(x). \tag{1}$$

且容易证明下面的结论成立:

(1) 给定两个集合 A 和 B，若 $A \subset B$，则 $S[A](x) \leqslant S[B](x)$;

(2) 对任意 g 和 f，$g \ll f$ 的充要条件是 $U[g] \subset U[f]$.

对于 $\forall (x, y), (h, k) \in U[f]$，将向量的点运算"$\cdot$"及逆元[3] 概念扩展到 $U[f]$ 中:

$$(x, y) \cdot (h, k) = (x \cdot h, y + k), \tag{2}$$

且 $(x, y) \in U[f]$ 的逆元为 $(\hat{x}, -y) \in \hat{U}[f]$.

同时，引入灰值图像函数 f 的旋转变换 $f_{h,k}$:

$$f_{h,k}(x) = f(x \cdot \hat{h}) + k = f((r_x, \theta_x) \cdot (r_h, \theta_h)) + k = f((r_x \cdot r_h, \theta_x + \theta_h)) + k, \tag{3}$$

其中 $\hat{h}$ 是 $h \in E^n$ 在旋转变换下的逆元[3]. 本影变换、表面算子与形态旋转变换之间具有如下性质.

性质 1　(1) $U[f_{h,k}] = (U[f]) \cdot (h, k)$; (2) $S_{h,k}(U[f])(x) = f_{h,k}(x)$.

证明　(1)
$U[f_{h,k}] = \{(x_1, y_1) \mid x_1 \in D[f_{h,k}], y_1 \leqslant f_{h,k}(x_1)\} =$
$\{(x_1, y_1) \mid x_1 \in D[f(x \cdot \hat{h}) + k], y_1 \leqslant f(x_1 \cdot \hat{h}) + k\} =$
$\{(x_1, y_1) \mid x_1 \cdot \hat{h} = x \in D[f], y_1 - k = y \leqslant f(x)\} =$
$\{(x \cdot h, y + k) \mid x \in D[f], y \leqslant f(x)\} =$
$\{(x, y) \cdot (h, k) \mid (x, y) \in U[f]\} = (U[f]) \cdot (h, k)$.

(2) $S_{h,k}(U[f])(x) = S(U[f])(x \cdot \hat{h}) + k = f(x \cdot \hat{h}) + k = f_{h,k}(x)$.

利用二值旋转形态算子的性质[3]，可建立本影变换、表面算子与极大极小运算及集合的交并运算之间的关系.

性质 2　(1) $U[f \wedge g] = U[f] \cap U[g]$;

(2) $S(U[f] \cap U[g]) = f \wedge g$;

(3) $U[f \vee g] = U[f] \cup U[g]$;

(4) $S(U[f] \cup U[g]) = f \vee g$.

证明　(1) $U[f \wedge g] = \{(x, y) \mid x \in D[f \wedge g], y \leqslant (f \wedge g)(x)\} =$
$\{(x, y) \mid x \in D[f] \cap D[g], y \leqslant \min\{f(x), g(x)\}\} =$
$\{(x, y) \mid x \in D[f], y \leqslant f(x); x \in D[g], y \leqslant g(x)\} =$
$U[f] \cap U[g]$;

(2) $S(U[f] \cap U[g]) = S(U[f \wedge g]) = f \wedge g$;

(3) $U[f \vee g] = \{(x, y) \mid x \in D[f]$ or $x \in D[g], y \leqslant \max\{f(x), g(x)\}\} =$
$\{(x, y) \mid x \in D[f], y \leqslant f(x)$ or $x \in D[g], y \leqslant g(x)\} = U[f] \cup U[g]$;

(4) $S[U[f] \cup U[g]] = S[U[f \vee g]] = f \vee g$.

3　灰值旋转形态算子的结构性质

利用本影变换和表面算子，在二值旋转形态算子[3] 的基础上引入结构元素 g 对于目标图像 f 的灰值旋转形态腐蚀、膨胀及开、闭算子:

$$\begin{cases} f\Theta_r g = S[U[f]\ \Theta_r U[g]], \\ f \oplus_r g = S[U[f] \oplus_r U[g]]. \end{cases} \tag{4}$$

$$\begin{cases} f \circ_r g = S(U[f] \circ_r U[g]), \\ f \bullet_r g = S(U[f] \bullet_r U[g]). \end{cases} \tag{5}$$

这里 $\Theta_r, \oplus_r, \circ_r, \bullet_r$ 表示二值及灰值旋转形态算子，以区别于平移形态算子. 利用性质 1 和性质 2，可以推出灰值旋转形态算子与灰值平移形态算子类似的表示及性质. 下面首先研究灰值旋转形态算子的结构特征，以下性质给出了灰值旋转算子的数学表达式.

性质 3　(1) $(f\Theta_r g)(x) = \min\{f(z \cdot x) - g(z) \mid z \in D[g], z \cdot x \in D[f]\}$;

(2) $(f \oplus_r g)(x) = \max\{f(x \cdot \hat{z}) + g(z) \mid z \in D[g], x \cdot \hat{z} \in D[f]\}$;

(3) $(f \circ_r g)(x) = \max\limits_{z}\{\min\limits_{t}\{f(x \cdot \hat{z} \cdot t) - g(t)\}$

$+ g(z) \mid t,z \in D(g), x \cdot \hat{z} \cdot t \in D(f)\}$;

(4) $(f \bullet_r g)(x) = \min\limits_z\{\max\limits_t\{f(x \cdot z \cdot \hat{t}) + g(t)\} - g(z) \mid z \in D(g), x \cdot z \cdot \hat{t} \in D(f)\}$.

证明 利用二值旋转形态腐蚀和膨胀的表达式[3]，有：

(1) $(f\Theta_r g)(x) = S[U[f]\Theta_r U[g]](x) =$
$\max\{y \mid (x,y) \in U[f]\Theta_r U[g]\} =$
$\max\{y \mid U[g]\cdot(x,y) \subseteq U[f]\} =$
$\max\{y \mid (z,w)\cdot(x,y) \in U[f], \forall z \in D[g], w \leqslant g(z)\} =$
$\max\{y \mid (z \cdot x, w+y) \in U[f], z \in D[g], w \leqslant g(z)\} =$
$\max\{y \mid w+y \leqslant f(z\cdot x), z \in D[g], w \leqslant g(z), z\cdot x \in D[f]\} =$
$\max\{y \mid y \leqslant f(z\cdot x) - g(z), z \in D[g], z\cdot x \in D[f]\} =$
$\min\{f(z\cdot x) - g(z) \mid z \in D[g], z\cdot x \in D[f]\}$;

(2) $(f\oplus_r g)(x) = S[U[f]\oplus_r U[g]](x) =$
$\max\{y \mid (x,y) \in U[f]\oplus_r U[g]\} =$
$\max\{y \mid (x,y)\cdot \hat{U}[g] \subseteq U[f]\} =$
$\max\{y \mid (x,y)\cdot(\hat{z},-w) \in U[f], (\hat{z},-w) \in \hat{U}[g], z \in D[g]\} =$
$\max\{y \mid (x\cdot\hat{z}, y-w) \in U[f], z \in D[g], w \leqslant g(z)\} =$
$\max\{y \mid y-w \leqslant f(x\cdot\hat{z}), z \in D[g], x\cdot\hat{z} \in D[f]\} =$
$\max\{y \mid y \leqslant f(x\cdot\hat{z}) + g(z), z \in D[g], x\cdot\hat{z} \in D[f]\} =$
$\max\{f(x\cdot\hat{z}) + g(z) \mid z \in D[g], x\cdot\hat{z} \in D[f]\}$;

(3) $(f\circ_r g)(x) = ((f\Theta_r g)\oplus_r g)(x) =$
$\max\limits_z\{(f\Theta_r g)(x\cdot\hat{z}) + g(z) \mid z \in D(g), x\cdot\hat{z} \in D(f)\} =$
$\max\limits_z\{\min\limits_t\{f(t\cdot x\cdot\hat{z}) - g(t)\} + g(z) \mid t,z \in D(g), t\cdot x\cdot\hat{z} \in D(f)\} =$
$\max\limits_z\{\min\limits_t\{f(x\cdot\hat{z}\cdot t) - g(t)\} + g(z) \mid t,z \in D(g), x\cdot\hat{z}\cdot t \in D(f)\}$;

(4) $(f\bullet_r g)(x) = (((f\oplus_r g)\Theta_r g)(x) =$
$\min\limits_z\{(f\oplus_r g)(z\cdot x) - g(z) \mid z \in D(g), z\cdot x \in D(f)\} =$
$\min\limits_z\{\max\limits_t\{f(z\cdot x\cdot\hat{t}) + g(t)\} - g(z) \mid z \in D(g), z\cdot x\cdot\hat{t} \in D(f)\} = \min\limits_z\{\max\limits_t\{f(x\cdot z\cdot\hat{t}) + g(t)\} - g(z) \mid z \in D(g), x\cdot z\cdot\hat{t} \in D(f)\}$.

性质4 (1) $U[f\Theta_r g] = U[f]\Theta_r U[g]$；

(2) $U[f\oplus_r g] = U[f]\oplus_r U[g]$.

证明 利用二值旋转形态腐蚀和膨胀的表示式[3]和性质3，有：

(1) $U[f\Theta_r g] = \{(x,y) \mid x \in D[f\Theta_r g], y \leqslant (f\Theta_r g)(x)\} =$
$\{(x,y) \mid x \in D[f] \cap D[g], y \leqslant f(z\cdot x) - g(z), \forall z \in D[g], z\cdot x \in D[f]\} =$
$\{(x,y) \mid x \in D[f] \cap D[g], y + g(z) \leqslant f(z\cdot x), z \in D[g], z\cdot x \in D[f]\} =$
$\{(x,y) \mid y + w \leqslant f(z\cdot x), z\cdot x \in D[f], w \leqslant g(z), z \in D[g]\} =$
$\{(x,y) \mid (z\cdot x, y+w) \in U[f], (z,w) \in U[g]\} =$
$\{(x,y) \mid (x,y)\cdot(z,w) \in U[f], (z,w) \in U[g]\} =$
$U[f]\Theta_r U[g]$；

(2) $U[f]\oplus_r U[g] = \{(x,y)\cdot(z,w) \mid (x,y) \in U[f], (z,w) \in U[g]\} =$
$\{(x\cdot z, y+w) \mid x \in D[f], y \leqslant f(x); z \in D[g], w \leqslant g(z)\} =$
$\{(t,s) \mid t = x\cdot z \in D[f]\oplus_r D[g], s = y + w \leqslant f(x) + g(z)\} =$
$\{(t,s) \mid t \in D[f\oplus_r g], s \leqslant \max\{f(t,\hat{z}) + g(z), z \in D[g], t\cdot\hat{z} \in D[f]\}\} =$
$\{(t,s) \mid t \in D[f\oplus_r g], s \leqslant (f\oplus_r g)(t)\} =$
$U[f\oplus_r g]$.

性质5 (1) $f\circ_r g = (f\Theta_r g)\oplus_r g$；

(2) $f\bullet_r g = (f\oplus_r g)\Theta_r g$.

证明 (1) 由二值旋转开运算的性质[3]及(5)式，有：

$f\circ_r g = S(U[f]\circ_r U[g]) =$
$S((U[f]\Theta_r U[g])\oplus_r U[g]) =$
$S(U[f\Theta_r g]\oplus_r U[g]) =$
$S(U[f\Theta_r g\oplus_r g]) = f\Theta_r g\oplus_r g$，

同理可证(2).

该性质表明，灰值旋转形态开、闭算子是灰值旋转形态腐蚀和膨胀算子的复合形式.

性质6

(1) $U[f\circ_r g] = U[f]\circ_r U[g]$；

(2) $U[f\bullet_r g] = U[f]\bullet_r U[g]$.

证明 利用性质4和性质5，得：

$U[f\circ_r g] = U[(f\Theta_r g)\oplus_r g] = U[f\Theta_r g]\oplus_r U[g] = (U[f]\Theta_r U[g])\oplus_r U[g] = U[f]\circ_r U[g]$，

同理可证(2).

下面的性质表明灰值旋转形态腐蚀、膨胀、开、闭算子满足对偶性.

性质7

(1) $f\Theta_r g = -[(-f)\oplus_r\hat{g}]$；

(2) $f\oplus_r g = -[(-f)\Theta_r\hat{g}]$；

(3) $f\circ_r g = -[(-f)\bullet_r\hat{g}]$；

(4) $f\bullet_r g = -[(-f)\circ_r\hat{g}]$，其中 $\hat{g}(x) = g(\hat{x})$.

证明 (1) $(f\Theta_r g)(x) =$
$\min\{f(z\cdot x) - g(z) \mid z \in D[g], z\cdot x \in D[f]\} =$
$-\max\{-f(z\cdot x) + g(z) \mid \forall z \in D[g], z\cdot x \in D[f]\} =$
$-\max\{-f(\hat{z}\cdot x) + g(\hat{z}) \mid \hat{z} \in D[g], \hat{z}\cdot x \in D[-f]\} =$
$-\max\{-f(\hat{z}\cdot x) + \hat{g}(z) \mid z \in D[\hat{g}], \hat{z}\cdot x \in D[-f]\} -$
$[(-f)\oplus_r\hat{g}](x)$；

(2) $(f\oplus_r g)(x) = \max\{f(x\cdot\hat{z}) + g(z) \mid z\in D[g],$
$x\cdot\hat{z}\in D[f]\}$ =
$-\min\{-f(x\cdot\hat{z}) - g(z) \mid z\in D[g], x\cdot\hat{z}\in D[f]\}$ =
$-\min\{-f(x\cdot z) - g(\hat{z}) \mid \hat{z}\in D[g], x\cdot z\in D[f]\}$ =
$-\min\{-f(x\cdot z) - \hat{g}(z) \mid z\in D[\hat{g}], x\cdot z\in D[-f]\}$ =
$-[(-f)\Theta_r\hat{g}](x)$;

(3) $f\circ_r g = (f\Theta_r g)\oplus_r g$ =
$-[(-(f\Theta_r g))\Theta_r\hat{g}]$ =
$-[(-(-[(-f)\oplus_r\hat{g}]))\Theta_r\hat{g}]$ =
$-[((-f)\oplus_r\hat{g})\Theta_r\hat{g}] = -[(-f)\bullet_r\hat{g}]$;

(4) $f\bullet_r g = (f\oplus_r g)\Theta_r g$ =
$-[(-(f\oplus_r g))\oplus_r\hat{g}]$ =
$-[(-(-[(-f)\Theta_r\hat{g}]))\oplus_r\hat{g}]$ =
$-[((-f)\Theta_r\hat{g})\oplus_r\hat{g}] = -[(-f)\circ_r\hat{g}]$.

性质 8

(1) $f\oplus_r g = g\oplus_r f$;

(2) $f\oplus_r(g\oplus_r h) = (f\oplus_r g)\oplus_r h$;

(3) $f\Theta_r(g\oplus_r h) = (f\Theta_r g)\Theta_r h$.

4　灰值旋转形态算子的代数性质

在研究了结构特征之后,进一步研究灰值旋转形态算子与其它相关运算之间的代数关系,首先,可以证明灰值旋转形态算子与极大、极小运算满足可交换性.

性质 9

(1) $(f\wedge g)\Theta_r h = (f\Theta_r h)\wedge(g\Theta_r h)$;

(2) $f\Theta_r(g\vee h) = (f\Theta_r g)\wedge(f\Theta_r h)$;

(3) $f\oplus_r(g\vee h) = (f\oplus_r g)\vee(f\oplus_r h)$.

同时,灰值旋转形态算子对于目标对象满足旋转不变性,即灰值旋转形态算子具有旋转不变性,是一类旋转不变算子.

性质 10

(1) $f_{h,k}\Theta_r g = (f\Theta_r g)_{h,k}$;

(2) $f_{h,k}\oplus_r g = (f\oplus_r g)_{h,k}$;

(3) $f_{h,k}\circ_r g = (f\circ_r g)_{h,k}$;

(4) $f_{h,k}\bullet_r g = (f\bullet_r g)_{h,k}$.

证明

(1) $f_{h,k}\Theta_r g = S[U[f_{h,k}]\Theta_r U[g]]$ =
$S[(U[f]\cdot(h,k))\Theta_r U[g]]$ =
$S[(U[f]\Theta_r U[g])\cdot(h,k)]$ =
$S[U[f\Theta_r g]\cdot(h,k)] = (f\Theta_r g)_{h,k}$;

(2) $f_{h,k}\oplus_r g = S[U[f_{h,k}]\oplus_r U[g]]$ =
$S[(U[f]\cdot(h,k))\oplus_r U[g]]$ =
$S[(U[f]\oplus_r U[g])\cdot(h,k)]$ =
$S[U[f\oplus_r g]\cdot(h,k)] = (f\oplus_r g)_{h,k}$;

(3) $f_{h,k}\circ_r g = (f_{h,k}\Theta_r g)\oplus_r g$ =
$(f\Theta_r g)_{h,k}\oplus_r g = ((f\Theta_r g)\oplus_r g)_{h,k}$ =
$(f\circ_r g)_{h,k}$;

(4) $f_{h,k}\bullet_r g = (f_{h,k}\oplus_r g)\Theta_r g = (f\oplus_r g)_{h,k}\Theta_r g$ =
$((f\oplus_r g)\Theta_r g)_{h,k} = (f\bullet_r g)_{h,k}$.

灰值旋转形态膨胀对于结构元素同样满足旋转不变性.结构元素的位置对灰值旋转形态腐蚀和膨胀的结果有影响,但不会对灰值旋转形态开、闭运算的结果产生影响.

性质 11

(1) $f\Theta_r g_{h,k} = (f\Theta_r g)_{\hat{h},-k}$;

(2) $f\oplus_r g_{h,k} = (f\oplus_r g)_{h,k}$;

(3) $f\circ_r g_{h,k} = f\circ_r g$;

(4) $f\bullet_r g_{h,k} = f\bullet_r g$.

证明　(1) $f\Theta_r g_{h,k} = S[U[f]\Theta_r U[g_{h,k}]]$ =
$S[U[f]\Theta_r(U[g]\cdot(h,k))]$ =
$S[(U[f]\Theta_r U[g])\cdot(\hat{h},-k)]$ =
$S[U[f\Theta_r g]\cdot(\hat{h},-k)] = (f\Theta_r g)_{\hat{h},-k}$;

(2) $f\oplus_r g_{h,k} = S[U[f]\oplus_r U[g_{h,k}]]$ =
$S[U[f]\oplus_r(U[g]\cdot(h,k))]$ =
$S[(U[f]\oplus_r U[g])\cdot(h,k)]$ =
$S[U[f\oplus_r g]\cdot(h,k)] = (f\oplus_r g)_{h,k}$;

(3) $f\circ_r g_{h,k} = (f\Theta_r g_{h,k})\oplus_r g_{h,k}$ =
$((f\Theta_r g_{h,k})\oplus_r g)_{h,k}$ =
$(((f\Theta_r g)_{\hat{h},-k})\oplus_r g)_{h,k}$ =
$(((f\Theta_r g)\oplus_r g)_{\hat{h},-k})_{h,k} = f\circ_r g$;

(4) $f\bullet_r g_{h,k} = (f\oplus_r g_{h,k})\Theta_r g_{h,k}$ =
$((f\oplus_r g_{h,k})\Theta_r g)_{\hat{h},-k}$ =
$(((f\oplus_r g)_{h,k})\Theta_r g)_{\hat{h},-k}$ =
$(((f\oplus_r g)\Theta_r g)_{h,k})_{\hat{h},-k} = f\bullet_r g$.

灰值旋转形态算子还具有相关的保序性:

性质 12　**如果**$f\ll g$,**则有:**

(1) $f\Theta_r h\ll g\Theta_r h$;

(2) $f\oplus_r h\ll g\oplus_r h$;

(3) $f\circ_r h\ll g\circ_r h$;

(4) $f\bullet_r h\ll g\bullet_r h$.

证明　**由**$f\ll g\Leftrightarrow U[f]\subset U[g]$,**及二值旋转形态腐蚀、膨胀的递增性,得:**

$U[f]\Theta_r U[h]\subset U[g]\Theta_r U[h]$,

$U[f]\oplus_r U[h]\subset U[g]\oplus_r U[h]$,**即**

$U[f\Theta_r h]\subset U[g\Theta_r h]$,

$U[f\oplus_r h]\subset U[g\oplus_r h]$,**故**

$$f\Theta_r h \ll g\Theta_r h, f\oplus_r h \ll g\oplus_r h;$$

进而有$(f\Theta_r h)\oplus_r h \ll (g\Theta_r h)\oplus_r h$，$(f\oplus_r h)\Theta_r h \ll (g\oplus_r h)\Theta_r h$，即：

$$f\circ_r h \ll g\circ_r h, f\bullet_r h \ll g\bullet_r h.$$

性质13 如果$g \ll h$，则有$f\Theta_r h \ll f\Theta_r g$，$f\oplus_r g \ll f\oplus_r h$.

证明 由$g \ll h \Leftrightarrow U[g] \subset U[h]$，利用二值旋转形态腐蚀、膨胀的相关性质得：

$$U[f]\Theta_r U[h] \subset U[f]\Theta_r U[g],$$

$$U[f]\oplus_r U[g] \subset U[f]\oplus_r U[h],$$

即$U[f\Theta_r h] \subset U[f\Theta_r g]$，$U[f\oplus_r g] \subset U[f\oplus_r h]$，故有：

$$f\Theta_r h \ll f\Theta_r g, f\oplus_r g \ll f\oplus_r h.$$

性质14 如果$U[g]$包含单位原点，则$f\Theta_r g \ll f \ll f\oplus_r g$.

证明 当$(1,0)\in U[g]$时，由二值旋转形态膨胀的扩展性和腐蚀的非扩展性及性质4知：

$$U[f] \subset U[f]\oplus_r U[g] = U[f\oplus_r g],$$

$$U[f\Theta_r g] = U[f]\Theta_r U[g] \subset U[f],$$

故有$f \ll f\oplus_r g$；$f\Theta_r g \ll f$.

由二值旋转形态开、闭运算的非扩展性、扩展性和幂等性，可证明灰值旋转形态开、闭运算具有类似的性质：

性质15 $f\circ_r g \ll f \ll f\bullet_r g$.

性质16 $(f\circ_r g)\circ_r g = f\circ_r g$，$(f\bullet_r g)\bullet_r g = f\bullet_r g$.

5 结语

本文在文献[3]的基础上，通过将点积运算扩展到本影集合，提出了本影集合中的点积和逆元方法，以此将平面旋转变换引入到函数中，建立了灰值图像函数的旋转变换，为利用本影变换和表面算子构造旋转形态算子提供了必备条件. 函数的旋转变换与本影和表面算子之间关系的建立为研究旋转形态算子提供了基础.

本文研究了灰值旋转形态腐蚀、膨胀、开、闭算子结构的建立及相关性质、方法，得出了有关算子表示、结构特征、代数性质等一系列重要结论，使得基于旋转变换的旋转形态学的理论和方法更加完善.

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