一类带有负指数的临界椭圆方程组的解

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摘 要 研究了一类带有强耦合临界非线性项和负指数项的椭圆方程组.定义了几个重要的约束集,运用复杂的分析技巧研究了能量泛函在约束集的下确界,得到了一个临界常数的精确表达式,最后证明了一定条件下方程组正解的存在性,首次把单个临界椭圆方程的相关结果推广到了带有负指数项的临界椭圆方程组.

关键词 椭圆方程组;临界非线性项;负指数项;变分法
中图分类号 O175.25 **文献标识码** A **文章编号** 1672-4321(2017)02-0143-05

Solutions to a Critical Elliptic System Involving Negative Exponents

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Abstract In this paper, a system of elliptic equations was investigated, which involves strongly-coupled critical nonlinearities and negative-exponent terms. Several constraint sets were defined, the infimums of the energy functional on the constraint sets were studied by complicated analytical techniques, and the explicit expression of a critical constant was obtained. Finally, the existence of positive solutions to the system was verified under certain conditions, and for the first time, the related conclusions for the single critical elliptic equation were extended to the system of critical elliptic equations involving negative-exponent terms.

Keywords elliptic system; critical nonlinearities; negative-exponent term; variational method

1 相关知识

在本文中,研究如下方程组:

$$\begin{cases} -\Delta u = \dfrac{\eta\alpha_1}{2^*}u^{\alpha_1-1}v^{\beta_1} + \dfrac{\alpha_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2-1}v^{\beta_2}, x\in\Omega, \\ -\Delta v = \dfrac{\eta\beta_1}{2^*}u^{\alpha_1}v^{\beta_1-1} + \dfrac{\beta_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2}v^{\beta_2-1}, x\in\Omega, \\ u,v>0, x\in\Omega; u=v=0, x\in\partial\Omega, \end{cases} \tag{1}$$

其中 $\Omega\subset\mathbb{R}^N(N\geqslant 3)$ 是有界光滑区域,$2^*:=\dfrac{2N}{N-2}$ 是临界 Sobolev 常数,有关常数满足如下假设条件:

(H_1) $\eta>0, \alpha_i, \beta_i>0, i=1,2, \alpha_1+\beta_1=2^*$, $\alpha_2+\beta_2<2, h(x)\in L^\infty(\Omega)$.

存在一个常数 α 满足 $\alpha\geqslant\max\{1-a_2, 1-\beta_2\}$, $m>0, M>0$,使得:
$m\cdot dist^\alpha(x,\partial\Omega)\leqslant h(x)\leqslant M\cdot dist^\alpha(x,\partial\Omega)$,

用 $H:=H_0^1(\Omega)$ 表示 $C_0^\infty(\Omega)$ 在范数 $(\int_\Omega|\nabla\cdot|^2dx)^{1/2}$ 下的完备化空间.方程组(1)在 $H\times H$ 中的能量泛函可以表示为:

$$J(u,v):=\frac{1}{2}\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx-\frac{\eta}{2^*}\int_\Omega|u|^{\alpha_1}\cdot|v|^{\beta_1}dx-\frac{1}{\alpha_2+\beta_2}\int_\Omega h(x)|u|^{\alpha_2}|v|^{\beta_2}dx.$$

由于 $\min\{\alpha_2,\beta_2\}\in(0,1)$,所以 $J\in C(H\times H,\mathbb{R})$.如果 $(u,v)\in H\times H$ 并且 $u,v>0$ a.e. 于 Ω 中,对所有 $(\varphi,\psi)\in H\times H$ 满足:

$$0=\int_\Omega(\nabla u\nabla\phi+\nabla v\nabla\psi-\frac{\eta\alpha_1}{2^*}u^{\alpha_1-1}v^{\beta_1}\phi-$$

收稿日期 2017-01-11
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基金项目 国家自然科学基金资助项目(11601530);中南民族大学研究生科研创新项目(2017sycxjj083)

$\frac{\eta\beta_1}{2^*}u^{\alpha_1}v^{\beta_1-1}\psi)\,dx - \int_\Omega(\frac{\alpha_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2-1}v^{\beta_2}\phi + \frac{\beta_2}{\alpha_2+\beta_2}h(x)u^{\alpha_2}v^{\beta_2-1}\psi)\,dx$, (2)

则称(u,v)是方程组(1)的一组解.

用$D:=D^{1,2}(\mathbb{R}^N)$表示$C_0^\infty(\mathbb{R}^N)$在范数$\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{1/2}$下的完备化空间. 对所有的$\alpha,\beta\in(0,2^*)$满足$\alpha+\beta=2^*$,基于Sobolev不等式和Young不等式,可以定义下面这两个最佳常数[1,2]:

$$\begin{cases} S:=\inf\limits_{u\in D\setminus\{0\}}\dfrac{\int_{\mathbb{R}^N}|\nabla u|^2dx}{(\int_{\mathbb{R}^N}|u|dx)^{2/2^*}}, \\ \tilde{S}(\alpha,\beta):=\inf\limits_{u,v\in D\setminus\{0\}}\dfrac{\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx}{(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta dx)^{2/2^*}}. \end{cases} \tag{3}$$

Talenti在文[2]中证明了Sobolev常数S有如下达到函数:

$$V_\varepsilon(x):=\frac{\varepsilon^{-(N-2)/2}(N(N-2))^{(N-2)/4}}{(1+|\varepsilon^{-1}x|^2)^{(N-2)/2}},\forall\varepsilon>0. \tag{4}$$

椭圆方程组已被很多学者研究过,其中u和v的指数通常是正的. 特别重要的是,Alves在文献[1]中获得了$\tilde{S}(\alpha,\beta)$和S的重要关系.

方程组(1)主要是受到如下椭圆问题的启发:

$$\begin{cases} -\Delta u=\dfrac{h(x)}{u^\gamma}+\lambda u^{p-1},x\in\Omega, \\ u>0,x\in\Omega;u=0,x\in\partial\Omega, \end{cases} \tag{5}$$

其中$\lambda\geqslant0,\gamma>0,1<p\leqslant2^*$,$h(x)$满足相应的条件. 方程(5)的解的存在性已经被很多学者研究过[3-6],因此研究对应的临界非线性椭圆方程组有非常重要的意义.

设$(u,v)\in H\times H$,定义$H\times H$上的范数$\|(u,v)\|^2:=\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx$.

因为$J\notin C^1(H\times H,\mathbb{R})$,定义如下的约束集:

$$\Lambda:=\{(u,v)=t(\tilde{u},\tilde{v})(\tilde{u},\tilde{v})\mid(\tilde{u},\tilde{v})\in(H\setminus\{0\})^2\}=\{(u,v)\in(H\setminus\{0\})^2\mid\Gamma(u,v)=0\},$$

其中$\Gamma(u,v)=\|(u,v)\|^2-\eta\int_\Omega|u|^{\alpha_1}|v|^{\beta_1}dx-\int_\Omega h(x)|u|^{\alpha_2}|v|^{\beta_2}dx$.

$t(\tilde{u},\tilde{v})>0$是下面映射的一个零点:

$$t\to\phi(t,\tilde{u},\tilde{v}):=\frac{1}{t^{2^*-1}}\frac{d}{dt}J(t\tilde{u},t\tilde{v})=t^{2-2^*}\|(\tilde{u},\tilde{v})\|^2-t^{\alpha_2+\beta_2-2^*}\int_\Omega h(x)|\tilde{u}|^{\alpha_2}|\tilde{v}|^{\beta_2}-\eta\int_\Omega|\tilde{u}|^{\alpha_1}|\tilde{v}|^{\beta_1}.$$

对任意的$(u,v)=t(\tilde{u},\tilde{v})(\tilde{u},\tilde{v})\in\Lambda$,定义:

$$I(u,v):=(2-2^*)\|(u,v)\|^2+(2^*-\alpha_2-\beta_2)\int_\Omega h(x)|u|^{\alpha_2}|v|^{\beta_2}dx.$$

基于$\frac{d}{dt}\phi(t,\tilde{u},\tilde{v})\Big|_{t=t(\tilde{u},\tilde{v})}$大于、等于或小于零,$\Lambda$的子集可以定义为[7]:

$\Lambda^+:=\{(u,v)\in\Lambda\mid I(u,v)>0\}$,$\Lambda^0:=\{(u,v)\in\Lambda\mid I(u,v)=0\}$,$\Lambda^-:=\{(u,v)\in\Lambda\mid I(u,v)<0\}$.

令$\eta^*:=\min\left\{\eta_1,\frac{\alpha_2+\beta_2}{2}\right\}$,其中:

$$\eta_1:=S^{\frac{2^*-\alpha_2-\beta_2}{2-\alpha_2-\beta_2}}\left(\frac{2-\alpha_2-\beta_2}{2^*-\alpha_2-\beta_2}\right)\left(\frac{2^*-2}{2^*-\alpha_2-\beta_2}\right)^{\frac{2^*-2}{2-\alpha_2-\beta_2}}\cdot\left(\left(\frac{\alpha_1}{\beta_1}\right)^{\frac{\beta_1}{2^*}}+\left(\frac{\beta_1}{\alpha_1}\right)^{\frac{\alpha_1}{2^*}}\right)^{\frac{2^*}{2}}\cdot\left(\frac{((\alpha_2/\beta_2)^{\frac{\beta_2}{\alpha_2+\beta_2}}+(\beta_2/\alpha_2)^{\frac{\alpha_2}{\alpha_2+\beta_2}})^{\frac{\alpha_2+\beta_2}{2}}}{\|h\|_\infty|\Omega|^{\frac{2^*-\alpha_2-\beta_2}{2^*}}}\right)^{\frac{2^*-2}{2-\alpha_2-\beta_2}}.$$

本文的主要结论如下.

定理1 假设条件(H_1)成立,并且$\eta<\eta_1$. 则方程组(1)至少有一个解.

在接下来的讨论中,先给出两个引理及其证明,之后再证明定理1. 当$t>0$,用$O(\varepsilon^t)$来表示满足$|O(\varepsilon^t)|/\varepsilon^t\leqslant C$的变量,这里$C$是正常数,用$o(\varepsilon^t)$表示满足当$\varepsilon\to0$时$|o(\varepsilon^t)|/\varepsilon^t\to0$的变量;为方便将省略积分式中的$dx$.

2 定理1的证明

对所有$(u,v)\in\Lambda$,有:

$$J(u,v)=\left(\frac{1}{2}-\frac{1}{2^*}\right)\|(u,v)\|^2-\left(\frac{1}{\alpha_2+\beta_2}-\frac{1}{2^*}\right)\int_\Omega h(x)|u|^{\alpha_2}|v|^{\beta_2}\geqslant C(\|(u,v)\|^2-\|(u,v)\|^{\alpha_2+\beta_2}).$$

这说明泛函J在Λ中是强制的并且有下界,因此可以考虑下面两个下确界问题:

$$\begin{cases} \tilde{d}:=\inf\limits_{(u,v)\in\Lambda}J(u,v), \\ \tilde{d}_\pm:=\inf\limits_{(u,v)\in\Lambda^\pm}J(u,v). \end{cases} \tag{6}$$

通过对$\Phi(t,u,v)$的性质的研究,发现$J(tu,tv)$在$[t^-(u,v),t^+(u,v)]$上是单调递增的,对所有的$(u,v)\in\Lambda^-$,有$t^+(u,v)=1$,并且:

$\tilde{d} \leqslant d_+ \leqslant J(t^-(u,v)u, t^-(u,v)v) \leqslant$

$J(t^+(u,v)u, t^+(u,v)v) \leqslant J(u,v)$，

所以 $\tilde{d} \leqslant d_+ \leqslant d_-$，运用文献 [8] 中引理 3 的证明方法，得到 $\Lambda^0 = \varnothing$，因此 $\tilde{d} = d_+$. 所以对于(6) 式，只需要考虑以下两个下确界：

$$\begin{cases} \tilde{d}_+ := \inf\limits_{(u,v)\in\Lambda^+} J(u,v), \\ \tilde{d}_- := \inf\limits_{(u,v)\in\Lambda^-} J(u,v). \end{cases} \tag{7}$$

为此选择满足下列两个性质的极小化序列 $\{(u_n, v_n)\} \subset \Lambda^+$，

$$\text{(i)}\ J(u_n,v_n) \leqslant \inf_{\Lambda^+} J(u,v) - \frac{1}{n}, \tag{8}$$

$$\text{(ii)}\ J(u,v) \geqslant J(u_n,v_n) - \frac{1}{n}\|(u-u_n, v-v_n)\|, \forall (u,v) \in \Lambda^+. \tag{9}$$

由于 $J(u,v) = J(|u|,|v|)$，所以 u_n, v_n 是径向对称的，假设 $u_0, v_0 \geqslant 0$. 因为 (u_n,v_n) 在 $H\times H$ 是有界的，所以 (u_n,v_n) 在 $H\times H$ 中弱收敛于 (u_0,v_0). 因此可以通过研究序列 $\{(u_n,v_n)\}$ 来获得有关 (u_0,v_0) 的性质，进一步证明 (u_0,v_0) 是问题(1) 的一个解.

为了叙述的方便，定义下面几个记号：

$G_i(u,v) = \alpha_i |u|^{\alpha_i-2}|v|^{\beta_i} u\varphi + \beta_i |u|^{\alpha_1}|v|^{\beta_i-2} v\psi, i = 1,2$，

$H(u,v) = \nabla u \nabla \varphi + \nabla v \nabla \psi$，

$F(u,v) = \frac{1}{t} h(x)(|u+t\varphi|^{\alpha_2}|v+t\psi|^{\beta_2} - |u|^{\alpha_2}|v|^{\beta_2})$，

其中 $\varphi, \psi \in C^\infty(\Omega)$.

引理 1　如果 $(\varphi,\psi) \in H\times H$，则下面的不等式成立：

$$\langle J(u_0,v_0), (\varphi,\psi)\rangle \geqslant 0. \tag{10}$$

证明　假设 $(\varphi,\psi) \in H\times H$ 并且 $\varphi,\psi > 0$，对每一个 (u_n,v_n) 都可以找到一个连续函数 $f_n(t)$，当 $t \geqslant 0$ 充分小的时候，使得 $f_n(t)(u_n + t\varphi, v_n + t\psi) \in \Lambda^+ (\subset \Lambda)$，显然，$f_n(0) = 1$. 事实上，重复文献 [8] 中引理 1 的步骤，易知 $f'_n(0)$ 关于 n 一致有界.

从序列 $\{(u_n,v_n)\}$ 的性质(ii) 可以得到：

$$\frac{1}{2}[f_n(t) - 1|\ \|(u_n,v_n)\| + t f_n(t)\|(\varphi,\psi)\|] \geqslant J(u_n,v_n) - J(f_n(t)(u_n+t\varphi), f_n(t)(v_n+t\psi)).$$

将上式除以 $t > 0$，并对变量 t 取极限，得到：

$$\frac{1}{n}[f'_n(0) - 1|\ \|(u_n,v_n)\| + \|(\varphi,\psi)\|] \geqslant \int_\Omega H(u_n,v_n) + \frac{\eta}{2^*}\int_\Omega G_1(u_n,v_n) + \liminf_{t\to 0}\frac{F(u_n,v_n)}{\alpha_2+\beta_2}.$$

因此：

$$\liminf_{t\to 0}\frac{F(u_n,v_n)}{\alpha_2+\beta_2} \leqslant \int_\Omega H(u_n,v_n) - \frac{\eta}{2^*}\int_\Omega G_1(u_n,v_n) + \frac{|f'_n(0)|\ \|(u_n,v_n)\| + \|(\varphi,\psi)\|}{n}.$$

注意到在 Ω 上 $F(u_n,v_n) \geqslant 0$，由 Fatou 引理知 $\liminf\limits_{t\to 0}\int_\Omega F(u_n,v_n)$ 是可积的，并且：

$$\int_\Omega \liminf_{t\to 0}\frac{F(u_n,v_n)}{\alpha_2+\beta_2} \leqslant \int_\Omega H(u_n,v_n) - \frac{\eta}{2^*}\int_\Omega G_1(u_n,v_n) + \frac{|f'_n(0)|\ \|(u_n,v_n)\| + \|(\varphi,\psi)\|}{n}.$$

又注意到：

$$\lim_{t\to 0}\frac{F(u_n,v_n)}{\alpha_2+\beta_2} = \begin{cases} 0, u_n = v_n = 0; \varphi,\psi \neq 0, \\ \infty, u_n \text{ 或 } v_n = 0; \varphi,\psi \neq 0, \\ \frac{1}{\alpha_2+\beta_2} h(x) G_2(u_n,v_n); u_n, v_n > 0; \varphi,\psi \neq 0, \end{cases}$$

假设 e_1 是方程 $\Delta e_1 + \lambda_1 e_1 = 0, x \in \Omega, e_1|_{\partial\Omega} = 0$ 的第一特征函数；取 $\varphi = \psi = e_1$ 作为检验函数，又 $u_n(x), v_n(x) > 0$，所以：

$$\int_\Omega \frac{h(x) G_2(u_n,v_n)}{\alpha_2+\beta_2} \leqslant \int_\Omega H(u_n,v_n) - \frac{\eta}{2^*}\int_\Omega G_1(u_n,v_n) + \frac{|f'_n(0)|\ \|(u_n+v_n)\| + \|(\varphi,\psi)\|}{n},$$

结合 $f'_n(0)$ 关于 n 一致有界，可以推知 $u_0(x) > 0$，$v_0(x) > 0$ a. e. 于 Ω，并且：

$$\int_\Omega \frac{1}{\alpha_2+\beta_2} h(x) G_2(u_0,v_0) \leqslant \int_\Omega H(u_0,v_0) - \frac{\eta}{2^*}\int_\Omega G_1(u_0,v_0), \tag{11}$$

因此引理 1 成立.

令：

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}, \varepsilon > 0, x \in \mathbb{R}^N,$$

是 Sobolev 不等式在 $\mathbb{R}^N$ 上的一个达到函数，定义截断函数 $\kappa(x) \in C_0^\infty(\Omega)$，$0 \leqslant \kappa(x) \leqslant 1$，当 $x \in B_r(0) \subset \Omega$，$\kappa(x) = 1$. 设 $U_{\varepsilon,a}(x) = \kappa(x) u_\varepsilon(x-a) \in H_0^1(\Omega)$. 显然有 $\|\nabla U_{\varepsilon,a}\|_2^2 = B + O(\varepsilon^{N-2})$，$\|U_{\varepsilon,a}\|_{2^*}^{2^*} = A + O(\varepsilon^N)$；$S = \frac{B}{A^{2/2^*}}$，$A$ 和 B 均为常数.

由文献 [1] 可知函数对：

$\left(\sqrt{\frac{\alpha}{2^*}} u_\varepsilon(x), \sqrt{\frac{\beta}{2^*}} u_\varepsilon(x)\right)$ 是 $\tilde{S}(\alpha,\beta)$ 的一个达

到函数,且:

$$\left\|\left(\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}(x),\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}(x)\right)\right\|^2 = B + O_1(\varepsilon^{N-2}),$$

$$\int_\Omega\left|\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}(x)\right|^{\alpha_1}\left|\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}(x)\right|^{\beta_1} = \tilde{A} + O_1(\varepsilon^N),$$

其中 $\tilde{A} = \left(\sqrt{\frac{\alpha_1}{2^*}}\right)^{\alpha_1}\left(\sqrt{\frac{\beta_1}{2^*}}\right)^{\beta_1}A = KA$,记 $\tilde{S} = \tilde{S}(\alpha,\beta)$,则有 $\tilde{S} = \frac{B}{\tilde{A}^{2/2^*}}$;为了方便叙述,令:

$$\tilde{G}_i(u_n,v_n) = \sqrt{\frac{\alpha_1}{2^*}}\alpha_i|u_n|^{\alpha_i-2}|v_n|^{\beta_i}u_n + \sqrt{\frac{\beta_1}{2^*}}\beta_i|u_n|^{\alpha_i}|v_n|^{\beta_i-2}v_n,$$

$$\tilde{H}(u_0,v_0) = \sqrt{\frac{\alpha_1}{2^*}}\nabla u_0\nabla U_{\varepsilon,a} + \sqrt{\frac{\beta_1}{2^*}}\nabla v_0\nabla U_{\varepsilon,a}.$$

引理2 如果 $\eta\in(0,\eta_1)$,则 $(u_0,v_0)\in\Lambda$.

证明 记 $a_0 = \Gamma(u_0,v_0)$,令 $u_0 = \varphi, v_0 = \psi$,由(11)式可知,$a_0\geqslant 0$,为了证明 $a_0 = 0$,不妨假设 $a_0 > 0$. 在该假设下,可以找到一个正数 $c_0 > 0$,使得:

$$c_0^2B - \eta c_0^{2^*}\tilde{A} = -a_0,\ \text{i.e.}$$
$$\tilde{S}(c_0\tilde{A}^{1/2^*})^2 - \eta(c_0\tilde{A}^{1/2^*})^{2^*} = a_0.$$

另一方面,有:

$$J(u_n,v_n)\to\mu_0 := \inf_{\Lambda^+}J(u,v),(u_n,v_n)\in\Lambda^+.$$

由文献[8],得到下面两个重要的结果,

(i) 可以找到一个常数 $c_{0,\varepsilon} = c_0 + \delta_\varepsilon$,其中 $\delta_\varepsilon\to 0$,当 $\varepsilon\to 0$ 时,使得:

$$\Gamma\left(u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}, v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right) = 0. \tag{12}$$

(ii) 下面这个等式成立:

$$\mu_0 = \left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)|u_0|^{\alpha_2}|v_0|^{\beta_2} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta\int_\Omega|u_0|^{\alpha_1}|v_0|^{\beta_1} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta c_0^{2^*}\tilde{A}.$$

由此可以推得,(u_0,v_0) 是下面泛函的局部极值函数对:

$$\left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)|u|^{\alpha_2}|v|^{\beta_2} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta\int_\Omega|u|^{\alpha_1}|v|^{\beta_1} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta c_{(u,v)}^{2^*}\tilde{A}. \tag{13}$$

对于泛函 $c_{(u,v)}$,令 $\varphi,\psi\in C_0^\infty(\Omega)$,取 $c_{(u,v)}$ 的估计函数为 $g(t) := c_{(u_0+t\varphi,v_0+t\psi)}$,这里 t 在零附近取值,因此有 $[g(t)]^2B - \eta[g(t)]^{2^*}\tilde{A} = -\Gamma(u_0+t\varphi,v_0+t\psi)$,又 $a_0 > 0$,并且 $g(0) = c_0$,所以:

$$\lim_{i\to 0}\{B[g(t)+g(0)] - 2^*\eta\tilde{A}[g(0)+\theta(g(t)-g(0))]^{2^*-1}]\}\frac{g(t)-g(0)}{t} = -[2\int_\Omega H(u_0,v_0) - \int_\Omega h(x)G_2(u_0,v_0) - \eta\int_\Omega G_1(u_0,v_0)].$$

这表明 $g'(0)$ 存在且:

$$g'(0) = -\frac{1}{M}[2\int_\Omega H(u_0,v_0) - \int_\Omega h(x)G_2(u_0,v_0) - \eta\int_\Omega G_1(u_0,v_0)], \tag{14}$$

其中 $M = 2c_0B - 2^*c_0^{2^*-1}\eta\tilde{A}$,考虑到(13)式,可得:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)|u_0+t\varphi|^{\alpha_2}|v_0+t\psi|^{\beta_2} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta\int_\Omega|u_0+t\varphi|^{\alpha_1}|v_0+t\psi|^{\beta_1} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta[g(t)]^{2^*}\tilde{A}\right\}\bigg|_{t=0} = 0,$$

推得:

$$\left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)G_2(u_0,v_0) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\eta\int_\Omega G_1(u_0,v_0) + 2^*\left(\frac{1}{2} - \frac{1}{2^*}\right)\eta c_0^{2^*-1}\tilde{A}\cdot\left\{-\frac{1}{M}\left[\int_\Omega(2H(u_0,v_0) - h(x)G_2(u_0,v_0) - \eta G_1(u_0,v_0))\right]\right\} = 0. \tag{15}$$

因此,可以由(15)式推得 $u_0,v_0\in C^{1,\beta}(\bar{\Omega})$,$\forall 0<\beta<1$. 在接下来的内容里,需要用到下面3个估计式[7,9]:

$$\int_\Omega\nabla u_0\nabla U_{\varepsilon,a} = O(\varepsilon^{(N-2)/2}), \tag{16}$$

$$\int_\Omega U_{\varepsilon,a}^{2^*-1}u_0 = \varepsilon^{(N-2)/2}\cdot u_0(a)\int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^{(N+2)/2}} + o(\varepsilon^{(N-2)/2}), \tag{17}$$

$$\int_\Omega|u_0|^{2^*-2}u_0U_{\varepsilon,a} = \varepsilon^{(N-2)/2}\int_{\mathbb{R}^N}\frac{|u_0|^{2^*-2}u_0\kappa(x)}{(|x-a|^2)^{(N-2)/2}} + o(\varepsilon^{(N-2)/2}). \tag{18}$$

由于 $h(x)|u_0|^{\alpha_2}|v_0|^{\beta_2}\in L^\infty(\Omega)$,利用中值定理,有:

$$\int_\Omega h(x)\left|u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}\right|^{\alpha_2}\left|v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right|^{\beta_2} = \int_\Omega h(x)|u_0|^{\alpha_2}|v_0|^{\beta_2} + \varepsilon^{(N-2)/2}\int_\Omega h(x)c_{0,\varepsilon}\tilde{G}_2(u_0,v_0)\frac{\kappa(x)}{(|x-a|^2)^{(N-2)/2}} + o(\varepsilon^{(N-2)/2}).$$

将上列估计式(16) ~ (18) 应用到(12) 式中得：

$$0 = \Gamma\left(u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}, v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right) = -[c_0^2 B - \eta c_0^{2^*}\tilde{A}] + c_{0,\varepsilon}^2 B - \eta c_{0,\varepsilon}^{2^*}\tilde{A} - \varepsilon^{(N-2)/2}\int_\Omega h(x)\frac{c_0\kappa(x)\tilde{G}_2(u_0,v_0)}{(|x-a|^2)^{(N-2)/2}} - \eta c_0\int_\Omega \tilde{G}_1(u_0,v_0)U_{\varepsilon,a} + 2c_0\int_\Omega \tilde{H}(u_0,v_0) + o(\varepsilon^{(N-2)/2}) - 2^*\eta c_0^{2^*-1}K\int_\Omega U_{\varepsilon,a}^{2^*-1}\left[\sqrt{\frac{\alpha_1}{2^*}}u_0 + \sqrt{\frac{\beta_1}{2^*}}v_0\right].$$

推得：

$$[M + o(1)]\cdot(-\delta_\varepsilon) = 2c_0\int_\Omega \tilde{H}(u_0,v_0) - c_0\int_\Omega h(x)\tilde{G}_2(u_0,v_0)U_{\varepsilon,a} - \eta c_0\int_\Omega \tilde{G}_1(u_0,v_0)U_{\varepsilon,a} - 2^*\eta c_0^{2^*-1}K\int_\Omega U_{\varepsilon,a}^{2^*-1}\left[\sqrt{\frac{\alpha_1}{2^*}}u_0 + \sqrt{\frac{\beta_1}{2^*}}v_0\right] + o(\varepsilon^{(N-2)/2}),$$

又由(15) 式推得：

$$(-\delta_\varepsilon) = \frac{\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}}{2^*\eta\left(\frac{1}{2} - \frac{1}{2^*}\right)c_0^{2^*-2}\tilde{A}}\int_\Omega h(x)\tilde{G}_2(u_0,v_0)U_{\varepsilon,a} + \frac{1}{2^*\tilde{A}c_0^{2^*-2}}\int_\Omega \tilde{G}_1(u_0,v_0)U_{\varepsilon,a} - \frac{2^*\eta c_0^{2^*-1}K}{M}\cdot\int_\Omega U_{\varepsilon,a}^{2^*-1}\left[\sqrt{\frac{\alpha_1}{2^*}}u_0 + \sqrt{\frac{\beta_1}{2^*}}v_0\right] + o(\varepsilon^{(N-2)/2}). \quad (19)$$

且知 $\delta_\varepsilon = O(\varepsilon^{(N-2)/2})$，由于 $a_0 > 0$，显然有：

$$M = \frac{2}{c_0}\left[c_0^2 - \eta\frac{2^*}{2}c_0^{2^*}\tilde{A}\right] < \frac{2}{c_0}[c_0^2 - \eta c_0^{2^*}\tilde{A}] = -\frac{2}{c_0}\cdot a_0 < 0. \quad (20)$$

又注意到：

$$\left(u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}, v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right) \in \Lambda,$$

结合(13) 和(19) 式可得：

$$J\left(u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}, v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right) = \left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)\left|u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}\right|^{\alpha_2}\left|v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right|^{\beta_2} + \eta\left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega\left|u_0 + c_{0,\varepsilon}\sqrt{\frac{\alpha_1}{2^*}}U_{\varepsilon,a}\right|^{\alpha_1}\left|v_0 + c_{0,\varepsilon}\sqrt{\frac{\beta_1}{2^*}}U_{\varepsilon,a}\right|^{\beta_1}$$

$$= u_0 + 2^*\eta\left(\frac{1}{2} - \frac{1}{2^*}\right)c_0^{2^*-1}\tilde{A}\left\{\frac{2^*\eta c_0^{2^*-1}K}{M}\cdot\int_\Omega U_{\varepsilon,a}^{2^*-1}\left[\sqrt{\frac{\alpha_1}{2^*}}u_0 + \sqrt{\frac{\beta_1}{2^*}}v_0\right] + (-\delta_\varepsilon)\right\} + 2^*\eta\left(\frac{1}{2} - \frac{1}{2^*}\right)c_0^{2^*-1}K\int_\Omega U_{\varepsilon,a}^{2^*-1}\left[\sqrt{\frac{\alpha_1}{2^*}}u_0 + \sqrt{\frac{\beta_1}{2^*}}v_0\right] + o(\varepsilon^{(N-2)/2}) = \mu_0 + 2^*\eta\left(\frac{1}{2} - \frac{1}{2^*}\right)c_0^{2^*-1}K\cdot\frac{2c_0B}{M} + o(\varepsilon^{(N-2)/2}) < \mu_0,$$

这是不可能的，因此 $a_0 = 0$，$(u_0, v_0) \in \Lambda$.

定理 1 的证明　令 $\varphi, \psi \in H$，$\varepsilon > 0$，定义：

$\Phi = (u_0 + \varepsilon\varphi)^+$，

$\Psi = (v_0 + \varepsilon\psi)^+$，$(\Phi, \Psi) \in H \times H$，

$\Omega_1 = \{x \mid u_0 + \varepsilon\varphi > 0\}$，

$\Omega_2 = \{x \mid v_0 + \varepsilon\psi > 0\}$，$\Omega^- = \Omega - \Omega_1 \cap \Omega$，

所以 $\Phi(x) = u_0 + \varepsilon\varphi|_{\Omega_1}$，$\Omega_2 = v_0 + \varepsilon\psi|_{\Omega_2}$，由引理 2 及不等式(11) 可得：

$$0 \leqslant \langle J(u_0,v_0),(\Phi,\Psi)\rangle \leqslant \varepsilon\left[\int_\Omega H(u_0,v_0) - \frac{\eta}{2^*}\int_\Omega G_1(u_0,v_0) - \frac{1}{\alpha_2+\beta_2}\int_\Omega h(x)G_2(u_0,v_0)\right] - \varepsilon\int_{\Omega^-}H(u_0,v_0), \quad (21)$$

显然，当 $\varepsilon \to 0$ 时，$\Omega^- = \{x \mid u_0 + \varepsilon\varphi \leqslant 0, v_0 + \varepsilon\psi \leqslant 0, x \in \Omega\}$ 的测度亦趋向于0，由此可知 $\int_\Omega(\nabla u_0\nabla\varphi + \nabla v_0\nabla\psi) \to 0$. 将(21) 式除以 ε 并对 ε 取极限，可得：

$$\int_\Omega H(u_0,v_0) - \frac{\eta}{2^*}\int_\Omega G_1(u_0,v_0) - \frac{1}{\alpha_2+\beta_2}\int_\Omega h(x)G_2(u_0,v_0) \geqslant 0.$$

因为对 $(-\varphi, -\psi)$ 上式同样成立，因此由引理 1 可知 (u_0, v_0) 是方程组(1) 的一个解，又知 (u_n, v_n) 在 $H \times H$ 中弱收敛于 (u_0, v_0)，引理 2 证得 $(u_0, v_0) \in \Lambda^+$，所以 $J(u_n, v_n) \to \inf_{\Lambda^+} J(u,v)$.

$$\inf_{(u,v)\in\Lambda^+} J(u,v) \geqslant \left(\frac{1}{2} - \frac{1}{2^*}\right)\|(u_0,v_0)\|^2 + \left(\frac{1}{2} - \frac{1}{\alpha_2+\beta_2}\right)\int_\Omega h(x)u_0^{\alpha_2}v_0^{\beta_2} = J(u_0,v_0),$$

所以 $J(u_0, v_0) = \inf_{\Lambda^+} J(u,v)$，定理 1 证毕.

(下转第 154 页)

Let $h(p) = 9p^2 - (8n+5)p - (8n+5)$ ($p \geqslant 4$, $n \geqslant p+2$), then if $h(p) = 0$, we have $p = \frac{(8n+5) + \sqrt{64n^2 - 208n - 155}}{18}$, if $n - 2 \geqslant p > \frac{(8n+5) + \sqrt{64n^2 - 208n - 155}}{18}$ the $R^*(G_3) > R^*(G_4)$; if $4 \leqslant p < \frac{(8n+5) + \sqrt{64n^2 - 208n - 155}}{18}$, then $R^*(G_3) < R^*(G_4)$. By repeated applications of the same method, when q is fixed, it is easy to see that the maximum multiplicative degree-Kirchhoff index, which is $\max\{R^*(P_n^{3,q}), R^*(P_n^{p,q})\}$. Note that:

$$R^*(P_n^{3,q}) - R^*(P_n^{p,q}) = \frac{1}{3}(p-3)[4(p+3)(n+1) - (3p^2+3p+9)] > 0 (\text{Since } p \geqslant 4).$$

Hence we have $R^*(P_n^{3,q}) > R^*(P_n^{p,q})$.

Similarly, by direct calculation, we have:

$$R^*(P_n^{3,3}) = \frac{2}{3}n^3 + 2n^2 - 23n + \frac{89}{3}, n \geqslant 5.$$

$$R^*(P_n^{3,3}) - R^*(P_n^{3,q}) = (\frac{4}{3}q^2 - 12)n - q^3 + \frac{4}{3}q^2 + 15 > 0 (\text{Since } q \geqslant 4).$$

Hence we get $R^*(P_n^{3,3}) > R^*(P_n^{3,q})$.

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(上接第 147 页)

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