第 36 卷第 2 期 中南民族大学学报(自然科学版) Vol. 36 No. 2
2017 年 6 月 Journal of South-Central University for Nationalities(Nat. Sci. Edition) Jun. 2017

Bicyclic Graphs with Extremal Multiplicative Degree-Kirchhoff Index

Zhu Zhongxun, Hong Yunchao, Luo Amu, Wang Weifeng*
(College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China)

Abstract For a graph G, the multiplicative degree-Kirchhoff index is defined as $R^*(G) = \sum_{\{x,y\} \subset V(G)} d_G(x) d_G(y) r_G(x, y)$. Based on the previous research results, similar to the additive degree-Kirchhoff index in the bicyclic graphs, the method of the same problem on the multiplicative degree-Kirchhoff index to the bicyclic graphs are discussed. First, some transformations on $R^*(G)$ are given, and then according to these transformations, the values of the minimum and maximum multiplicative degree-Kirchhoff index of bicyclic graphs with n-vertex having precisely two cycles are obtained and the corresponding extremal graphs are characterized. The degree-Kirchhoff index is widely used in the fields of electrical network, chemistry, Markov chain and Euclidean distance, and so on.

Keywords distance; multiplicative degree-Kirchhoff index; bicyclic graphs
中图分类号 O157 **文献标识码** A **文章编号** 1672-4321(2017)02-0148-07

具有乘积度-基尔霍夫指标极值的双圈图

朱忠熏, 洪运朝*, 罗阿木, 王维峰
(中南民族大学 数学与统计学学院, 武汉 430074)

摘　要　对于一个图 G, 乘积度-基尔霍夫指标定义为 $R^*(G) = \sum_{\{x,y\} \subset V(G)} d_G(x) d_G(y) r_G(x, y)$. 基于前人的一些研究成果, 用类似于和的度-基尔霍夫指标应用在双圈图中的方法, 把乘积度-基尔霍夫指标运用到双圈图中. 首先给出了关于 $R^*(G)$ 的一些图变换, 然后根据这些图变换, 确定了恰好有两个圈的 n 阶双圈图的最小和最大的乘积度-基尔霍夫指标的值及其对应的极值图. 度-基尔霍夫指标广泛应用于电流网络、化学、马尔可夫链和欧氏距离等各个方面.
关键词　距离; 乘积度-基尔霍夫指标; 双圈图

Let $G = (V(G), E(G))$ be a connected simple graph. The distance $d_G(x, y)$ is defined as the length of a shortest path between vertices x and y in G. Suppose that G_1 and G_2 are two disjoint connected graphs with $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, let $(G_1, u_1) \oplus (G_2, u_2)$ be the graph created by coalescence of vertices u_1 and u_2. A graph G is called bicyclic graph if $|E(G)| = |V(G)| + 1$ [1,2].

Let $\beta_n^{p,q}$ be the class of bicyclic graphs with exactly two cycles $C_p = v_1 v_2 \cdots v_p v_1$ and $C_q = u_1 u_2 \cdots u_q u_1$, which is connected by a path $P = v_1 w_1 \cdots w_{m-1} u_1$. Let T_{v_i}, T_{u_j} and T_{w_k} be trees rooted at v_i, u_j and w_k, respectively, as shown in Fig. 1.

We say a tree T is trivial if $|V(T)| = 1$. If $v_1 (= u_1)$ is the unique common vertex of C_p and C_q, let $S_n^{p,q}$ be the graph obtained from cycles C_p and C_q by attaching $n + 1 - p - q$ pendent edges to v_1. Let $P_n^{p,q}$ be the graph obtained from C_p and C_q, which is connected by a path of length $n - p - q + 1$, as shown in Fig. 2. The Kirchhoff index $Kf(G)$ [3,4], proposed by Klein et

收稿日期　2016-11-17　* 通讯作者　洪运朝, 研究方向: 图论; E-mail: 331963706@qq.com
作者简介　朱忠熏(1973-), 男, 副教授, 博士, 研究方向: 图论; E-mail: zzxun73@mail.scuec.edu.cn
基金项目　国家自然科学基金资助项目(61070197); 中南民族大学中央高校基本科研业务费专项资金资助项目(CZQ10007)

al. is defined as $Kf(G) = \sum_{\{x,y\} \subset V(G)} r_G(x, y)$. The additive degree-Kirchhoff index $R^+(G)$, pointed by I. Gutman et al[3], is defined as

$$R^+(G) = \sum_{\{x,y\} \subset V(G)} (d_G(x) + d_G(y)) r_G(x,y).$$

The multiplicative degree-Kirchhoff index, introduced by Chen and Zhang in Ref [6], is defined as $R^*(G) = \sum_{\{x,y\} \subset V(G)} d_G(x) d_G(y) r_G(x,y)$.

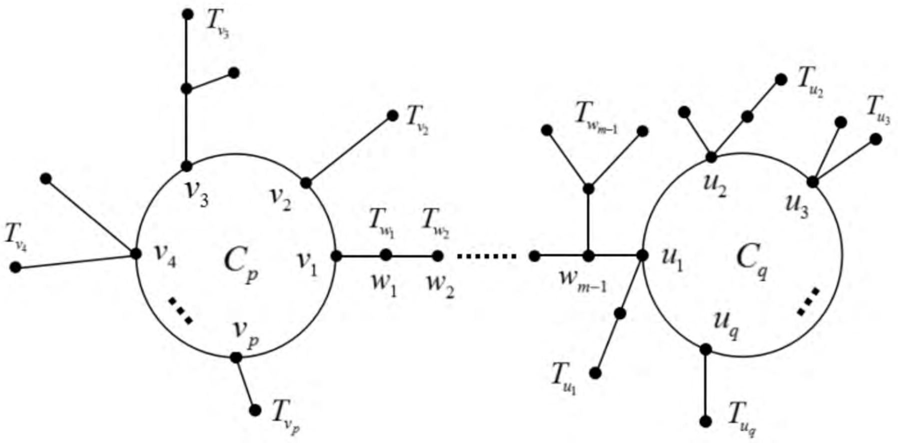

Fig. 1　Graph $\beta_n^{p,q}$

图1　图 $\beta_n^{p,q}$

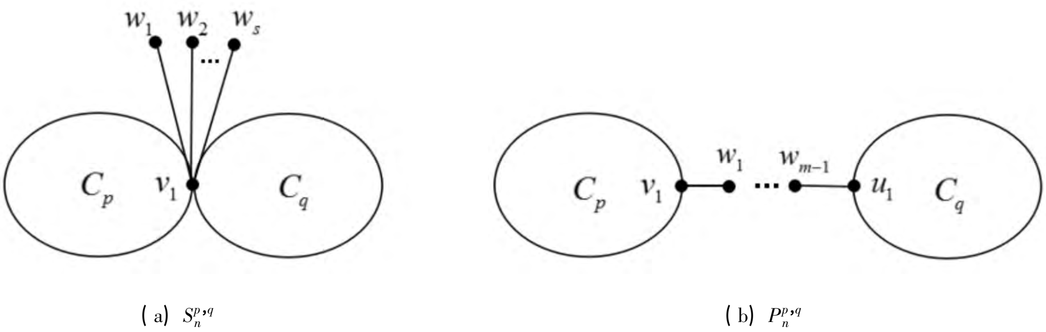

Fig. 2　Graphs $S_n^{p,q}$ and $P_n^{p,q}$

图2　图 $S_n^{p,q}$ 和 $P_n^{p,q}$

In this paper, we will study the upper bound of multiplicative degree-Kirchhoff index of graphs in $\beta_n^{p,q}$ and characterize the corresponding extremal graphs.

The following lemmas that will be used in the proof of our main results.

Lemma 1[4]　Let u be a cut vertex of a connected graph G, x and y be two vertices in different components of $G-u$, then $r_G(x,y) = r_G(x,u) + r_G(u,y)$.

Lemma 2[7,8]　Let P_n, C_n and S_n be the path, the cycle and the star on n vertices, respectively. Then

$$R^*(P_n) = (n-1)^2 + \frac{(n-1)(n-2)(2n-3)}{3},$$

$$R^*(C_n) = \frac{n^3 - n}{3},$$

$$R^*(S_n) = (n-1)(2n-3).$$

Lemma 3[7,8]　Let G_1 and G_2 be connected graphs with disjoint vertex sets, with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$. Constructing the graph G by identifying the vertices u_1 and u_2, and denote the so obtained vertex by u. Then $R^*(G) = R^*(G_1) + R^*(G_2) + 2m_2 \sum_{x \in V(G_1)} d_{G_1}(x) \cdot r_{G_1}(u,x) + 2m_1 \sum_{y \in V(G_2)} d_{G_2}(y) r_{G_2}(u,y)$.

1　Some Transformations

In this section, we will give some transformations which will decrease or increase $R^*(G)$.

Transformation 1　Let u_1u_2 be a cut-edge of bicyclic graph G, C_p, C_q be the connected components of $G - u_1u_2$, where $u_1 \in V(C_p)$, $u_2 \in V(C_q)$.

Constructing the graph G^* from G by deleting u_1u_2 and identifying the vertices u_1, u_2, denote the so obtained vertex by u, adding an pendent edge uv.

Lemma 4 Let G, G^* be the graphs described in Transformation 1, then $R^*(G) > R^*(G^*)$.

Proof Let $|V(C_p)| = p$, $|V(C_q)| = q$, and $|E(C_p)| = p$, $|E(C_q)| = q$. Let $H = G[V(G) \setminus (V(C_q) - u_2)]$, $H^* = G[V(G^*) \setminus (V(C_q) - u)]$.

By Lemma 3, we have:

$$R^*(G) = R^*(H) + R^*(C_q) + 2q\sum_{x\in V(H)} d_H(x) r_H(u_2, x) + 2(p+1)\sum_{y\in V(G_2)} d_{C_q}(y) r_{C_q}(u_2, y).$$

$$R^*(G^*) = R^*(H^*) + R^*(C_q) + 2q\sum_{x\in V(H)} d_H(x) r_H(u, x) + 2(p+1)\sum_{y\in V(G_2)} d_{C_q}(y) r_{C_q}(u, y).$$

Then $R^*(G) - R^*(G^*) = 2q\left[\sum_{x\in V(H)} d_H(x) r_H(u_2, x) - \sum_{x\in V(H^*)} d_{H^*}(x) r_{H^*}(u, x)\right] = 4pq > 0$. Hence $R^*(G) > R^*(G^*)$.

Let β_n be the class of connected graphs on n vertices. By Transformation 1 and Lemma 4, we have the following result.

Corollary 1 Let G_0 be a graph with the smallest multiplicative degree-Kirchhoff index in β_n, then all cut-edges are pendent edges.

Transformation 2 Let G be a bicyclic graph with $V(G) = \{u, v, v_1, v_2, \cdots, v_s\} \cup V(C_p) \cup V(C_q)$, for which v is a vertex of degree $s+1$ such that $vv_1, vv_2, \cdots, vv_s$ are pendent edges incident with v, and u is the neighbor of v distinct from v_i that is on the cycle C_q. The other cycle C_p only has one common vertex w with C_q. We form a graph $G' = \sigma(G, v)$ by deleting the edges $vv_1, vv_2, \cdots, vv_s$ and adding new edges $uv_1, uv_2, \cdots, uv_s$. We say that G' is a σ-transform of the graph G.

Lemma 5 Let G and G' be the graphs defined in Transformation 2. Then $R^*(G) > R^*(G')$.

Proof Let $T = G[\{v, v_1, v_2, \cdots, v_s\}]$, $H = G[V(G) \setminus V(T)]$, then

$$R^*(G) = \frac{1}{2}\sum_{x,y\in V(H)-u} d_G(x) d_G(y) r_G(x, y) + \frac{1}{2}\sum_{x,y\in V(T)-v} d_G(x) d_G(y) r_G(x, y) + \sum_{x\in V(H)-u, y\in V(T)-v} d_G(x) d_G(y) r_G(x, y) + \sum_{x\in V(H)-u} d_G(x) d_G(u) r_G(x, u) + \sum_{y\in V(T)-v} d_G(v) d_G(y) r_G(v, y) + \sum_{x\in V(H)-u} d_G(x) d_G(v) r_G(x, v) + \sum_{y\in V(T)-v} d_G(u) d_G(y) r_G(u, y) + d_G(u) d_G(v) r_G(u, v).$$

$$R^*(G') = \frac{1}{2}\sum_{x,y\in V(H)-u} d_{G'}(x) d_{G'}(y) r_{G'}(x, y) + \frac{1}{2}\sum_{x,y\in V(T)-v} d_{G'}(x) d_{G'}(y) r_{G'}(x, y) + \sum_{x\in V(H)-u, y\in V(T)-v} d_{G'}(x) d_{G'}(y) r_{G'}(x, y) + \sum_{x\in V(H)-u} d_{G'}(x) d_{G'}(u) r_{G'}(x, u) + \sum_{y\in V(T)-v} d_{G'}(v) d_{G'}(y) r_{G'}(v, y) + \sum_{x\in V(H)-u} d_{G'}(x) d_{G'}(v) r_{G'}(x, v) + \sum_{y\in V(T)-v} d_{G'}(u) d_{G'}(y) r_{G'}(u, y) + d_{G'}(u) d_{G'}(v) r_{G'}(u, v).$$

Hence we have $R^*(G) - R^*(G') = 2s\left[\sum_{x\in V(H)-u} d_G(x) + d_G(u) - 1\right]$. Note that $V(H) - u \neq \Phi$ and $d_G(u) > 1$. So $R^*(G) > R^*(G')$.

Lemma 6 Let G_0 be a bicyclic graph G with $V(G) = \{v_1, v_2, \cdots, v_s\} \cup V(C_p) \cup V(C_q)$, for which u is a vertex of degree $s+2$ in the cycle C_q of the bicyclic graph G_0, and $uv_1, uv_2, \cdots, uv_s$ are pendent edges incident with u, and the other cycle C_p only has one common vertex w with C_q. Let graph G_1 delete the edges $uv_1, uv_2, \cdots, uv_s$, and add new edges $wv_1, wv_2, \cdots, wv_s$. Then $R^*(G_0) > R^*(G_1)$.

Proof Let $G = G_0[V(C_p) \cup V(C_q)]$, $H_0 = G[V(G_0) \setminus (V(G) - u)]$, and $H_1 = G[V(G_1) \setminus (V(G) - w)]$, then $H_0 \cong H_1 \cong K_{1,s}$. By Lemma 3, we have:

$$R^*(G_0) = R^*(G) + R^*(K_{1,s}) + 2s\sum_{x\in V(G)} d_G(x) r_G(u, x) + 2|E(G)|\sum_{y\in V(K_{1,s})} d_{K_{1,s}}(y) r_{K_{1,s}}(u, y).$$

$$R^*(G_1) = R^*(G) + R^*(K_{1,s}) + 2s\sum_{x\in V(G)} d_G(x) r_G(w, x) + 2|E(G)|\sum_{y\in V(K_{1,s})} d_{K_{1,s}}(y) r_{K_{1,s}}(w, y).$$

Hence we get:

$$R^*(G_0) - R^*(G_1) = 2s\left[\sum_{x\in V(G)} d_G(x) r_G(u, x) - \sum_{x\in V(G)} d_G(x) r_G(w, x)\right] = 2s\left[\sum_{x\in V(C_p)} d_G(x) r_G(u, w) + d_G(w) r_G(u, w)\right] > 0.$$

Then we have $R^*(G_0) > R^*(G_1)$.

Transformation 3 Let G be a bicyclic graph with $V(G) = \{u, v, v_1, v_2, \cdots, v_s\} \cup V(C_p) \cup V(C_q)$, for which v is a vertex of degree $s+1$ such that vv_1, vv_2, $\cdots$, vv_s are pendent edges incident with v, and u is the neighbor of v distinct from v_i that is on the cycle C_q. The other cycle C_p only has one common vertex w with C_q. We form a graph $G'' = \pi(G, v)$ by deleting the edges $vv_1, vv_2, \cdots, vv_s$ and connecting v_i and all the isolated vertices into a path $vv_1v_2\cdots v_s$.

We say that G'' is a π-transform of the graph G.

Lemma 7 Let G and G'' be the graphs defined in Transformation 3. Then $R^*(G) < R^*(G'')$.

Proof Let $T = G[\{v, v_1, v_2, \cdots, v_s\}]$, $H = G[V(G) \setminus V(T)]$, $R^*(G)$ is defined in Lemma5, then

$$R^*(G'') = \frac{1}{2}\sum_{x,y \in V(H)-u} d_{G''}(x)d_{G''}(y)r_{G''}(x,y) + \frac{1}{2}\sum_{x,y \in V(T)-v} d_{G''}(x)d_{G''}(y)r_{G''}(x,y) + \sum_{x \in V(H)-u, y \in V(T)-v} d_{G''}(x)d_{G''}(y)r_{G''}(x,y) + \sum_{x \in V(H)-u} d_{G''}(x)d_{G''}(u)r_{G''}(x,u) + \sum_{y \in V(T)-v} d_{G''}(v)d_{G''}(y)r_{G''}(v,y) + \sum_{x \in V(H)-u} d_{G''}(x)d_{G''}(v)r_{G''}(x,v) + \sum_{y \in V(T)-v} d_{G''}(u)d_{G''}(y)r_{G''}(u,y) + d_{G''}(u)d_{G''}(v)r_{G''}(u,v).$$

Hence we have:

$$R^*(G) - R^*(G'') = -\left[(s^2+1)\sum_{x \in V(H)-u} d_G(x) + s(s-1)d_G(u) + \frac{(s-1)(s-2)(2s-3)}{3}\right].$$

Since $V(H) - u \neq \Phi$, therefore we have $d_G(x) > 0$, $d_G(u) > 1$. Then we have $R^*(G) - R^*(G'') < 0$. This completes the proof.

Lemma 8 Let G_0 be a bicyclic graph with the vertex set $V(C_p) \cup V(C_q) \cup V(P_{s+1})$, in which $V(C_p) \cap V(P_{s+1}) = \{v\}$ and $V(C_q) \cap V(P_{s+1}) = \{w\}$. For $wa \in E(P_{s+1})$, and $u \in V(C_q)$, let $G_1 = (G_0 - \{aw\}) \cup \{ua\}$, then $R^*(G_0) > R^*(G_1)$.

Proof Let H, H_0 and H_1 be the graphs as shown in Fig. 3, then we have

$$G_0 = (H, v_{s-1}) \oplus (H_0, a),$$
$$G_1 = (H, v_{s-1}) \oplus (H_1, w).$$

By Lemma 3, we have:

$$R^*(G_0) = R^*(H) + R^*(H_0) + 2(q+1) \cdot \sum_{x \in V(H)} d_H(x)r_H(a,x) + 2(p+s-1)\sum_{y \in V(H_0)} d_{H_0}(y)d_{H_0}(a,y).$$

$$R^*(G_1) = R^*(H) + R^*(H_1) + 2(q+1) \cdot \sum_{x \in V(H)} d_H(x)r_H(w,x) + 2(p+s-1)\sum_{y \in V(H_1)} d_{H_1}(y)d_{H_1}(w,y).$$

Hence $R^*(G_0) - R^*(G_1) = 4(p+s-1)[q - r_{C_q}(w,u)] \geqslant 0$. If $q = r_{C_q}(w,u)$, then w and u coincidence, G_0 and G_1 is isomorphic. Since G_0 and G_1 is not isomorphic, therefore we get $R^*(G_0) - R^*(G_1) > 0$. This completes the proof.

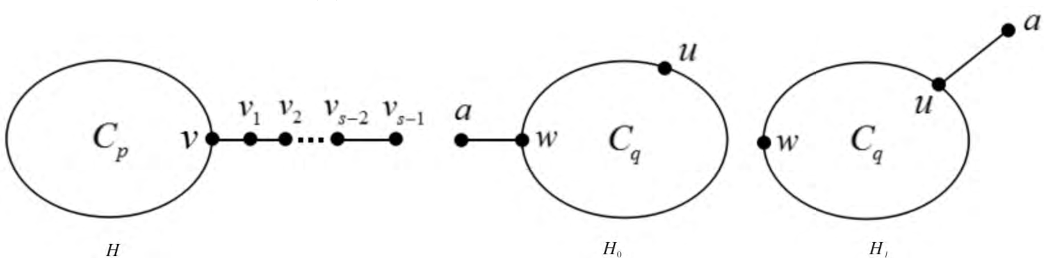

Fig. 3 Graphs H, H_0 and H_1

图 3 图 H, H_0 和 H_1

2 Main results

In this section, we will characterize n-vertex bicyclic graphs with exactly two cycles having the minimum and maximum multiplicative degree-Kirchhoff index.

Theorem 1 Let $G \in \beta_n^{p,q}$, $G \neq S_n^{p,q}$. Then $R^*(G) > R^*(S_n^{p,q})$.

Proof Suppose that a bicyclic graph G_0 has minimal multiplicative degree-Kirchhoff index among graphs in $\beta_n^{p,q}$. For G_0, we prove the following results.

(1) In Fig. 1, T_{v_i}, T_{u_j} and T_{w_k} are all stars with their centers at v_i, u_j and w_k for each i, j and k.

Without loss of generality, suppose that tree T_{v_i} is not a star. Let G_1 be constructed from G_0 by deleting all the edges of T_{v_i} and connecting all the isolated vertices to v_i. By Lemma 5, we have $R^*(G_0) > R^*(G_1)$, which contradicts the choice of G_0. Hence (1) holds.

(2) The length of the path connects the two cycles in G_0 is zero.

Suppose that there exist the length of path is k ($k \geqslant 1$) in G_0. Assume that $v_1 = w_0$, $u_1 = w_k$. Let $e = w_i w_{i+1}$ be an edge of path. Let G_2 be the graph obtained from G_0 by first contracting e and then attaching a pendent edge $w_i a$ to w_i. Assume that G_{01} and G_{02} are two components of $G_0 - e$ and G_{21} and G_{22} are copies of G_{01} and G_{02} in G_2, respectively. See Fig. 4.

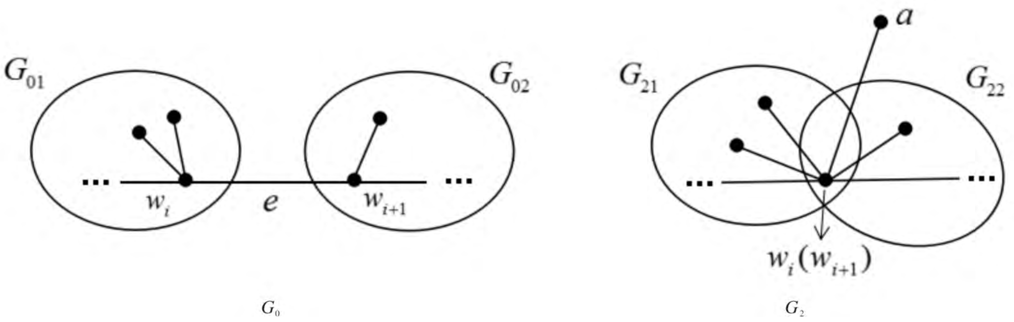

Fig. 4 Graphs G_0 and G_2

图4 图 G_0 和 G_2

By Lemma 4 and Corollary 1, we have $R^*(G_0) > R^*(G_2)$. This contradicts the hypothesis. Hence (2) holds.

(3) In Fig. 1, if $p + q \leqslant n$, then only T_{v_1} ($T_{v_1} = T_{u_1}$) is nontrivial.

Without loss of generality, suppose to the contrary that tree T_{u_i} ($i \neq 1$) is nontrivial. By Lemma 6, we get $R^*(G_0) > R^*(G_1)$, which contradicts the choice of G_0. Hence (3) holds.

According to (1)-(3), we get Theorem 1.

Theorem 2 Let $G \in \beta_n^{p,q}$, $G \neq P_n^{p,q}$. Then $R^*(G) < R^*(P_n^{p,q})$.

Proof Suppose that a bicyclic graph G_0 has maximal multiplicative degree-Kirchhoff index among graphs in $\beta_n^{p,q}$. For G_0, we prove the following claims.

(1) In Fig. 1, T_{v_i}, T_{u_j} and T_{w_k} are all paths with their centers at v_i, u_j and w_k for each i, j and k.

Without loss of generality, suppose that tree T_{v_i} is not a path. Let G_1 be constructed from G_0 by deleting all the edges of T_{v_i} and connecting all the isolated vertices into a path. By Lemma 7, we have $R^*(G_1) > R^*(G_0)$, which contradicts the choice of G_0. Hence (1) holds.

(2) Assume that $T_{w_0} = T_{v_1}$ and $T_{w_m} = T_{u_1}$, then T_{w_i} is trivial $0 \leqslant i \leqslant m$).

If not, without loss of generality, suppose that there is nontrivial T_{w_j}. By (1), we know that T_{w_j} is a path with w_j as its end vertex and assume that u is the other end vertex. Let $G_2 = G_0 - w_j w_{j+1} + u w_{j+1}$ (if $j = m$, $G_2 = G_0 - w_{j-1} w_j + u w_{j-1}$). Assume that G_{01} and G_{02} are two components of $G_0 - w_j w_{j+1}$ and G_{21} and G_{22} are two components of $G_2 - u w_{j+1}$. See Fig. 5.

In the following, we prove $R^*(G_2) > R^*(G_0)$.

Let $H_0 = G_{02} + w_j w_{j+1}$, $H_2 = G_{22} + u w_{j+1}$, $r_G(w_j, u) = s$, then by Lemma 3, we get:

$$R^*(G_0) - R^*(G_2) = 2(q+1)\Big[\sum_{x \in V(G_{01})} d_{G_{01}}(x)\, r_{G_{01}}(w_j, x) - \sum_{x \in V(G_{21})} d_{G_{21}}(x)\, r_{G_{21}}(u, x)\Big] = -4(q+1)ps \leqslant 0 \text{ (Since } p \geqslant 3, q \geqslant 3, s \geqslant 0).$$

If $s = 0$, then G_0 and G_2 is isomorphic. Since G_0 and G_2 is non-isomorphic. Therefore we obtain $R^*(G_2) > R^*(G_0)$.

This contradicts the hypothesis. Hence (2) holds.

(3) In Fig. 1, if $p + q \leqslant n$, then T_{v_i} and T_{u_j} are trivial for each i and j.

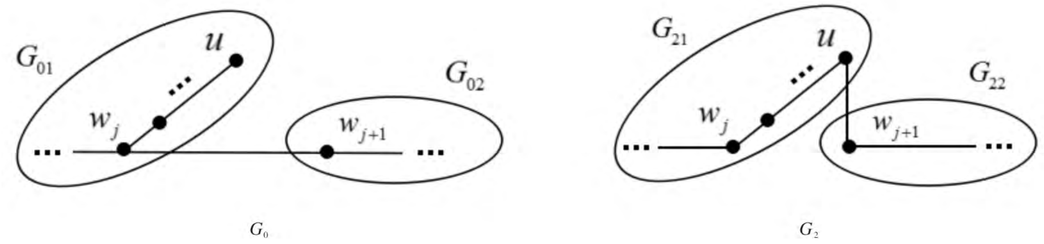

Fig. 5　Graphs G_0 and G_2

图 5　图 G_0 和 G_2

Without loss of generality, suppose to the contrary that tree $T_{v_i}(i \neq 1)$ is nontrivial. By Lemma 8, we get $R^*(G_0) > R^*(G_1)$, which contradicts the choice of G_0. Hence (3) holds.

According to (1)-(3), we get Theorem 2.

By direct calculation, we have:

$$R^*(S_n^{p,q}) = 2n^2 + \left[\frac{2}{3}(p^2+q^2) - 2(p+q) + \frac{5}{3}\right]n - \frac{1}{3}\left[(p^3+q^3) - 2(p^2+q^2) + 2(p+q) + 1\right].$$

$$R^*(P_n^{p,q}) = \frac{2}{3}n^3 + 2n^2 - \left[\frac{4}{3}(p^2+q^2) - 1\right]n + \left[(p^3+q^3) - \frac{4}{3}(p^2+q^2) - \frac{1}{3}\right].$$

3　Bicyclic graphs with extremal multiplicative degree-Kirchhoff index

By Theorem 1 and Theorem 2, the graphs in $\beta_n^{p,q}$ with minimum and maximum multiplicative degree-Kirchhoff index must belong to the classes of $S_n^{p,q}$ and $P_n^{p,q}$, respectively. In what follows, we will determine those which has the extremal multiplicative degree-Kirchhoff index among in $\beta_n^{p,q}$.

Theorem 3　Among all n-vertex bicyclic graphs, the graph $S_n^{3,3}$ has the minimum multiplicative degree-Kirchhoff index.

$$R^*(S_n^{3,3}) = 2n^2 + \frac{5}{3}n - \frac{31}{3},\quad n \geq 5.$$

Proof　Let u_1, u_2, w be three successive vertices lying on the C_p of the bicyclic graph G_1. The other cycle C_q only has one common vertex w with C_p. And $wv_1, wv_2, \cdots, wv_s$ are pendent edges incident with w.

Let the graph G_2 is obtained by deleting the edges u_1u_2 and adding the edge wu_2. Then $R^*(G_2) < R^*(G_1)$.

Let $H_1 = G[V(G_1) \setminus (V(C_q) - w)]$, $H_2 = G[V(G_2) \setminus (V(C_q) - w)]$, then

$$\begin{aligned} R^*(G_1) - R^*(G_2) = {} & [R^*(C_p) - R^*(C_{p-1})] + [R^*(S_{s+1}) - R^*(S_{s+2})] + \Big[2s\sum_{x\in V(C_p)} d_{C_p}(x) r_{C_p}(w,x) - \\ & 2(s+1)\sum_{x\in V(C_{p-1})} d_{C_{p-1}} r_{C_{p-1}}(w,x)\Big] + \\ & \Big[2p\sum_{y\in V(S_{s+1})} d_{S_{s+1}}(y) r_{S_{s+1}}(w,y) - 2(p-1)\cdot \sum_{y\in V(S_{s+2})} d_{S_{s+2}}(y) r_{S_{s+2}}(w,y)\Big] + \Big[2q\sum_{x\in V(H_1)} d_{H_1}(x)\cdot \\ & r_{H_1}(w,x) - 2q\sum_{x\in V(H_2)} d_{H_2}(x) r_{H_2}(w,x)\Big] = \\ & \frac{4}{3}pq + \frac{4}{3}s(p-2) + \frac{1}{3}\left[\left(p - \frac{7}{2}\right)^2 - \frac{69}{4}\right] > 0 \end{aligned}$$

(Since $p \geq 4, q \geq 3, s \geq 0$).

Combining the above discussions, according to $R^*(S_n^{p,q})$ and $p = q = 3$, we can get the minimum multiplicative degree-Kirchhoff index.

$$R^*(S_n^{3,3}) = 2n^2 + \frac{5}{3}n - \frac{31}{3}, n \geq 5.$$

Theorem 4　Among all n-vertex bicyclic graphs, the graph $P_n^{3,3}$ has the maximum multiplicative degree-Kirchhoff index.

$$R^*(P_n^{3,3}) = \frac{2}{3}n^3 + 2n^2 - 23n + \frac{89}{3}, n \geq 5.$$

Proof　Let u_1, w, u_2 be three successive vertices lying on the C_p of the bicyclic graph G_3. The cycle C_p and C_q are linked with two end vertices v and w of P_{s+1}. Let the graph G_4 is obtained by deleting the edge wu_2 and adding the edge u_1u_2. Then $R^*(G_4) > R^*(G_3)$.

Let $H_3 = G[V(G_3) \setminus (V(C_p) - w)]$, $H_4 = G[V(G_4) \setminus (V(H_3) - w)]$. Then

$$\begin{aligned} R^*(G_3) - R^*(G_4) = {} & [R^*(C_p) - R^*(H_4)] + 2(q+s)\Big[\sum_{x\in V(C_p)} d_{C_p}(x) r_{C_p}(w,x) - \sum_{x\in V(H_4)} d_{H_4}(x) r_{H_4}(w,x)\Big] \\ = {} & \frac{1}{3}\left[9p^2 - (8n+5)p - (8n+5)\right]. \end{aligned}$$

Let $h(p)=9p^2-(8n+5)p-(8n+5)$ ($p\geqslant 4$, $n\geqslant p+2$), then if $h(p)=0$, we have $p=\frac{(8n+5)+\sqrt{64n^2-208n-155}}{18}$, if $n-2\geqslant p>\frac{(8n+5)+\sqrt{64n^2-208n-155}}{18}$ the $R^*(G_3)>R^*(G_4)$; if $4\leqslant p<\frac{(8n+5)+\sqrt{64n^2-208n-155}}{18}$, then $R^*(G_3)<R^*(G_4)$. By repeated applications of the same method, when q is fixed, it is easy to see that the maximum multiplicative degree-Kirchhoff index, which is $\max\{R^*(P_n^{3,q}), R^*(P_n^{p,q})\}$. Note that:

$R^*(P_n^{3,q})-R^*(P_n^{p,q})=\frac{1}{3}(p-3)[4(p+3)(n+1)-(3p^2+3p+9)]>0$ (Since $p\geqslant 4$).

Hence we have $R^*(P_n^{3,q})>R^*(P_n^{p,q})$.

Similarly, by direct calculation, we have:

$R^*(P_n^{3,3})=\frac{2}{3}n^3+2n^2-23n+\frac{89}{3}, n\geqslant 5.$

$R^*(P_n^{3,3})-R^*(P_n^{3,q})=(\frac{4}{3}q^2-12)n-q^3+\frac{4}{3}q^2+15>0$ (Since $q\geqslant 4$).

Hence we get $R^*(P_n^{3,3})>R^*(P_n^{3,q})$.

References

[1] Bondy J A, Murty U S R. Graph theory with applications [M]. New York: Macmillan, 1976: 16-93.

[2] Liu J B, Zhang S Q, Pan X F, et al. Bicyclic graphs with extremal degree resistance distance [EB/OL]. (2016-01-03) [2016-12-28]. http://www.arXiv:1606.01281v1.com/2016/01/03/bicyclic-graphs-with-extremal-degree-resistance-distance/.

[3] Bonchev D, Balaban A T, Liu X, et al. Molecular cyclicity and centricity of polyclic graphs I cyclicity based on resistance distances or reciprocal distances [J]. Int J Quantum Chem, 1994, 50: 1-20.

[4] Klein D J, Randic M. Resistance distance [J]. J Math Chem, 1993, 12: 81-95.

[5] Gutman I, Feng L. Degree resistance distance of unicyclic graphs [J]. Trans Comb 1, 2012, 1: 27-40.

[6] Chen H Y, Zhang F J. Resistance distance and the normalized Laplacian spectrum [J]. Discr Appl Math, 2007, 155: 654-661.

[7] Palacios J L, Renom J M. Another look at the degree-Kirchhoff index [J]. Int J Quantum Chem, 2011, 111: 3453-3455.

[8] Palacios J L. Upper and lower bounds for the additive degree-Kirchhoff index [J]. Match Commun Math Comput Chem, 2013, 70: 651-655.

(上接第147页)

参 考 文 献

[1] Alves C, Filho D, Souto M. On systems of elliptic equations involving subcritical or critical Sobolev exponents [J]. Nonlinear Analysis, 2000, 42: 771-787.

[2] Talenti G. Best constant in Sobolev inequality [J]. Annali di Matematica Pura ed Applicata, 1976, 110(1): 353-372.

[3] Chen Y, Chen J. Existence of multiple positive weak solutions and estimates forextremal values to a class of elliptic problems with Hardy terms and singular nonlinearity [J]. Journal of Mathematical Analysis and Applications, 2015, 492: 873-900.

[4] Chen J, Rocha E. Positive solutions for elliptic problems with critical nonlinearity and combined singularity [J]. Mathematica Bohemica, 2010, 135: 413-422.

[5] Lazer A, Mckenna P. On a singular nonlinear elliptic boundary value problem [J]. Proceedings of the American Mathematical Society, 1991, 111: 720-730.

[6] Sun Y, Wu S, Long Y. Combined effects of singular and superlinear nonlinearities in some singular boundary value problems [J]. Journal of Differential Equations, 2001, 176: 511-531.

[7] Tarantello G. On nonhomogenous elliptic equations involving critical Sobolev exponent [J]. Annales de l'Institut Henri Poincare Analyse Non Lineaire, 1992, 9(3): 281-304.

[8] Sun Y, Zhang D. The role of the power 3 for elliptic equations with negative exponents [J]. Calculus Variations Partial Differential Equations, 2014, 49: 909-922.

[9] 康东升,黄 燕,刘 殊. 一类拟线性椭圆问题极值函数的渐近估计 [J]. 中南民族大学学报(自然科学版), 2008, 27(3): 91-95.