第36卷第4期　　中南民族大学学报(自然科学版)　　Vol. 36 No. 4
2017年12月　　Journal of South-Central University for Nationalities(Nat. Sci. Edition)　　Dec. 2017

带消失位势 Choquard 方程解的存在性

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摘　要　研究了一类非局部 Schrödinger 方程解的存在性.运用山路引理和 Ekland 变分法,利用泛函几何结构和极小化序列得到了方程解的存在性.首次将半线性椭圆方程的相关结果推广到 Choquard 型消失位势 Schrödinger 方程.

关键词　Choquard 方程; 山路引理; Ekland 变分法; 消失位势

中图分类号　O175.25　**文献标识码**　A　**文章编号**　1672-4321(2017)04-0149-03

Existence of Solutions of Choquard Type Equations with Vanishing Potentials

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Abstract　In this paper, a class of nonlocal Schrödinger equations were studied. The Mountain Pass Lemma and Ekland variational principle were used to prove the existence of solutions. Some new results about Choquard type Schrödinger equations with vanishing potentials were obtained by related results in the semilinear elliptical equations.

Keywords　Choquard type equations; Mountain Pass Lemma; Ekland variational principle; vanishing potentials

1　主要问题及结果

本文考虑一类带消失位势 Choquard 方程:

$$-\Delta u+V(x)u=\left(\int_{R^N}\frac{|u(y)|^p}{|x-y|^{(N-\alpha)}}\mathrm{d}y\right)|u|^{p-2}u+K(x)f(u),x\in R^N \tag{1}$$

解的存在性,其中 $N\geqslant 3,\alpha\in(0,N),p\in\left(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}\right)$,$V(x)$,$K(x)$,$f$ 满足(文[1,2]):

(i) $V(x),K(x)>0,\forall x\in R^N,\lim\limits_{|x|\to\infty}V(x)=0$,$K\in L^\infty(R^N)$.

(ii) 如果 $\{A_n\}\subset R^N$ 是一个 Borel 集合列,且存在 $M>0$,对所有的 n 都有 $\|A_n\|\leqslant M$,那么 $\lim\limits_{r\to+\infty}\int_{A_n\cap B_r^c}K(x)\,\mathrm{d}x=0$ 关于 n 一致成立.

(iii) $\frac{K}{V}\in L^\infty(R^N)$.

(iv) 存在某个 q,$q\in(2,2^*)$,使得 $\frac{K}{V^{\frac{2^*-q}{2^*-2}}}\to 0$,$x\to+\infty$.

(H_1) $\limsup\limits_{t\to 0}\frac{f(t)}{t}=0$,如果(iii)成立;$\limsup\limits_{t\to 0}\frac{f(t)}{t^{q-1}}=0$,如果(iv)成立.

(H_2) $\limsup\limits_{t\to 0}\frac{f(t)}{t^{2^*-1}}=0$.

(H_3) $t^{-1}f(t)$,$t\in(0,+\infty)$ 是非降函数,且 $\limsup\limits_{t\to+\infty}\frac{F(t)}{t^2}=+\infty$,其中 $F(t)=\int_0^t f(t)\,\mathrm{d}t$.

(H_4) 存在 $2p>\theta>2$ 使得 $0\leqslant\theta F(t)\leqslant tf(t)$,对任意 $t\in R$ 都成立,其中 $F(t)=\int_0^t f(t)\,\mathrm{d}t$.

收稿日期　2017-09-12

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基金项目　中央高校基本科研业务费专项资金资助项目(CZQ12014)

Choquard型方程解的存在性的研究,近年来一直吸引着研究者的兴趣. 文[3]中Moroz和Schaftingen考虑了基态解的存在性,文[4]讨论该类方程解的存在集中性,其他相关解的存在性问题见文[5-7]. 本文的主要结果是定理1.

定理1 假设条件(i)~(iv),(H_1)~(H_4)成立,那么方程(1)在$D^{1,2}(R^N)$中至少存在两个非平凡解.

2 预备知识

令$E=\left\{u\in D^{1,2}(R^N):\int_{R^N}V(x)u^2\mathrm{d}x<+\infty\right\}$.

定义其上的内积为$(u,v)_E=\int_{R^N}\nabla u\nabla v+V(x)uv\mathrm{d}x$,由它所诱导的范数是:

$$\|u\|=\left(\int_{R^N}|\nabla u|^2+V(x)u^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

令$L_K(R^N)=\{u:R^N\to R,u$是可测的,且$\int_{R^N}K(x)|u|^p\mathrm{d}x<+\infty\}$.

记$|u|_{L^s(\Omega)}=\left(\int_\Omega|u|^s\mathrm{d}x\right)^{\frac{1}{s}}$,特别地$\Omega=R^N$,记$|u|_{L^s}=|u|_{L^s(R^N)}=\int_{R^N}|u|^s\mathrm{d}x$.

与方程组(1)所对应的能量泛函可以写为:

$$I(u)=\frac{1}{2}\|u\|^2-\frac{1}{2p}Q(u,u)-\int_{R^N}K(x)F(u)\mathrm{d}x,$$

其中

$$Q(u,v)=\int_{R^N}\int_{R^N}\frac{|u(y)|^p|u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y.$$

由Hardy-Littlewood不等式[8]知:

当$p\in\left(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}\right)$,$u\in L^{\frac{2Np}{N+\alpha}}(R^N)$时:

$$Q(u,u)=\int_{R^N}\int_{R^N}\frac{|u(y)|^p|u(x)|^p}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y\le C(N,\alpha,s,t)|u|_{L^s}^p|u|_{L^t}^p<+\infty. \quad (2)$$

其中C为依赖于N,α,s,t的常数且$\frac{1}{s}+\frac{1}{t}+\frac{N-\alpha}{N}=1$. 同时由此可得$I\in C^1(E,R)$.

当$u\in E$满足:

$$\langle I'(u),v\rangle=\int_{R^N}\nabla u\nabla v+V(x)uv\mathrm{d}x-Q(u,v)-\int_{R^N}K(x)f(u)v\mathrm{d}x,$$

其中$v\in E$,则称u为I在E上的一个弱解.

设E为一实的希尔伯特空间,泛函$I\in C^1(E,R)$. 我们说$\{u_n\}$为I的P.S.序列:如果当$n\to\infty$时,有$I(u_n)\to c$,$I'(u_n)\to0$. 泛函I在指标$c\in R$处满足P.S.条件是指,上述$\{u_n\}$存在一个收敛的子序列. 若$I(u)=c$,$I'(u)=0$时,称u为I在E上的临界点,c为I的临界值.

引理1 $I\in C^1(E,R)$且I满足:

(1) 存在$\beta,\rho>0$使得$I(u)\ge\beta$对任意$\|u\|=\rho$时都成立;

(2) 存在$e\in E$,$\|e\|>\rho$且使得$I(e)<0$.

证明 (1) $I(u)\ge\frac{1}{2}\|u\|^2-\frac{1}{2p}|Q(u,u)|-\int_{R^N}|K(x)||F(u)|\mathrm{d}x\ge\frac{1}{2}\|u\|^2-\frac{1}{2p}C(N,\alpha,s,t)|u|_{L^s}^p|u|_{L^t}^p-|K|_{L^\infty}|u|_{L^q}^q\ge\frac{1}{2}\|u\|^2-C\|u\|^{2p}-|K|_{L^\infty}\|u\|^q$.

由此可知,存在足够小的$\rho>0$使得$\|u\|=\rho$时:

$$I(u)\ge\frac{1}{2}\|u\|^2-C\|u\|^{2p}-|K|_{L^\infty}\|u\|^q\ge\frac{1}{2}\rho^2(1-2C\rho^{2p-2}-2|K|_{L^\infty}\rho^{q-2})>0.$$

令$\beta=\frac{1}{2}\rho^2(1-2C\rho^{2p-2}-2|K|_{L^\infty}\rho^{q-2})$,即证(1)成立;

(2) $I(tu)=\frac{t^2}{2}\|u\|^2-\frac{1}{2p}Q(tu,tu)-\int_{R^N}K(x)F(tu)\mathrm{d}x\le\frac{t^2}{2}\|u\|^2-\frac{1}{2p}Q(tu,tu)=\frac{t^2}{2}\|u\|^2-\frac{t^{2p}}{2p}Q(u,u)$.

由此可知当$t\to+\infty$时,$I(tu)<0$成立,故证存在$e\in E$,$\|e\|>\rho$且使得$I(e)<0$.

3 定理1的证明

在这一部分将证明主要结果定理1.

引理2 如果(i)~(iv),(H_1)~(H_4)成立,则有I满足P.S.条件.

证明 1) 任取$\{u_n\}\subset E$满足$I(u_n)\to c$并且$I'(u_n)\to0$,那么$\{u_n\}$有界.

事实上:

$$c(1+\|u_n\|)\ge I(u_n)-\frac{1}{\theta}(I'(u_n),u_n)=\left(\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2-\left(\frac{1}{2p}-\frac{1}{\theta}\right)Q(u_n,u_n)\right)+$$

$\int_{R^N} f(u_n)u_n - \theta F(u_n)\,dx \geqslant \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \int_{R^N} f(u_n)u_n - \theta F(u_n)\,dx.$

由条件(H_4)可知存在 c_1,使得:

$$\int_{R^N} f(u_n)u_n - \theta F(u_n)\,dx \geqslant - c_1,$$

因此:

$c(1+\|u\|) \geqslant k\|u\|^2 - c_1$,可知$\{u_n\}$有界.

2) $\{u_n\}$有界. 因此存在 $u \in E$,使得在 E 中,$u_n \xrightarrow{弱} u$,且在 $L_{loc}^{\frac{2Np}{N+\alpha}}(R^N)$ 中,$u_n \xrightarrow{强} u$. 又

$\|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u\rangle - \int_{R^N} K(x)(f(u_n) - f(u))(u_n - u)\,dx - Q(u_n - u, u_n - u)\,dx.$

由文[1,2],有:

$$\lim_{n\to\infty}\int_{R^N} K(x)f(u_n)u_n dx = \lim_{n\to\infty}\int_{R^N} K(x)f(u)u dx, \quad (3)$$

$$\lim_{n\to\infty}\int_{R^N} K(x)f(u_n)u dx = \lim_{n\to\infty}\int_{R^N} K(x)f(u)u dx. \quad (4)$$

$n\to\infty$ 时,

$$\lim_{n\to\infty}\int_{R^N} K(x)(f(u_n) - f(u))(u_n - u)\,dx = 0. \quad (5)$$

由文[9]有:

$$\lim_{n\to\infty} Q(u_n - u, u_n - u) = 0. \quad (6)$$

这里当 $n\to\infty$ 时,

$$\langle I'(u) - I'(u_n), u_n - u\rangle \to 0. \quad (7)$$

由(5)~(7)式可知 $n\to\infty$ 时,$\|u_n - u\| \to 0$.

定理1的证明　由山路引理及引理1的(1)及引理2知,存在能量泛函 I 的 P.S. 序列$\{u_n\}$,满足$\{u_n\}\subset E$,在 E 中有 $u_n \xrightarrow{强} u_1$,$\langle I'(u_1), v\rangle = 0$,$I(u_1) = \sigma > 0$,其中:

$\sigma = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I(\gamma(t))$,$\Gamma = \{\gamma(t) \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = e\}$.

又由引理1的(2)及引理2知,存在 $-\infty < \inf_{u\in B_\rho} I(u) < 0$ 由 Ekland 变分原理[10]知,存在极小化序列$\{v_n\}\subset B_\rho$,使得 $I(v_n) \to \inf_{u\in B_\rho} I(u)$,$I'(v_n) \to 0$. 在 E 中,$v_n \xrightarrow{强} u_2$,$I(u_2) = \inf_{u\in B_\rho} I(u)$,$\langle I'(u_2), v\rangle = 0$.

综上可知,方程(1)在 $D^{1,2}(R^N)$ 中至少存在两个非平凡解 u_1,u_2.

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