带有 Rellich 项的临界双调和方程组的研究

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摘　要　在全空间中研究了一类带有 Rellich 项的临界双调和方程组,得到了方程组的基态解.在有界区域上研究了另一类带有 Rellich 项和线性扰动项的临界双调和方程组,运用变分法证明了方程组在一定条件下存在非平凡解,首次把单个奇异双调和方程的相关结果推广到对应的方程组.

关键词　双调和;临界;方程组;非平凡解

中图分类号　O175.25　**文献标识码**　A　**文章编号**　1672-4321(2018)01-0149-05

On the Critical Biharmonic Equations Involving the Rellich-Type Terms

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Abstract　In this paper, a class of critical biharmonic equations involving the Rellich-type terms were investigated in R^N and the ground state solutions are obtained. Another class of critical biharmonic equations was investigated in a bounded domain, which involves the Rellich-type terms and linear perturbations. By variational arguments, the existence of nontrivial solutions to the problem was established under certain conditions, and for the first time, the related conclusions for the single singular biharmonic equation were extended to the system of corresponding equations.

Keywords　biharmonic; critical; equations; nontrivial solution

1　相关知识

首先在全空间中研究下面带有 Rellich 项的临界双调和方程组:

$$\begin{cases}\Delta^2 u-\mu\dfrac{u}{|x|^4}=\dfrac{u^{2^*(s)-1}}{|x|^s}+\dfrac{\eta\alpha}{2^*(s)}\dfrac{u^{\alpha-1}v^{\beta}}{|x|^s},\\ \Delta^2 v-\mu\dfrac{v}{|x|^4}=\dfrac{v^{2^*(s)-1}}{|x|^s}+\dfrac{\eta\beta}{2^*(s)}\dfrac{u^{\alpha}v^{\beta-1}}{|x|^s},\\ u,v>0,x\in R^N\setminus\{0\},\end{cases}\tag{1}$$

其中参数满足下列条件:(H_1) $N\geqslant 5,0\leqslant s<4,0<\eta<+\infty,\alpha,\beta>1,\alpha+\beta=2^*(s):=\dfrac{2(N-s)}{N-4},0\leqslant\mu<\bar{\mu}.2^*(s)$ 是临界指数,$\bar{\mu}:=\left(\dfrac{N(N-4)}{4}\right)^2$ 是下列 Rellich 不等式中的最佳常数:

$$\int_{R^N}\frac{u^2}{|x|^4}\mathrm{d}x\leqslant\frac{1}{\bar{\mu}}\int_{R^N}|\Delta u|^2\mathrm{d}x,\forall u\in D^{2,2}(R^N),\tag{2}$$

这里 $D^{2,2}(R^N)$ 表示 $C_0^\infty(R^N)$ 关于$\left(\int_{R^N}|\Delta\cdot|^2\mathrm{d}x\right)^{\frac{1}{2}}$ 的完备化空间,令 $D:=(D^{2,2}(R^N)\setminus\{0\})^2$,根据 Rellich 不等式我们得到算子 $L_\mu[\,]:=\Delta^2\cdot-\mu\dfrac{\cdot}{|x|^4}$ 是正定的.进一步,存在一个常数 $C(s)>0$ 使得下面的 Rellich-Sobolev 不等式成立:

$$\left(\int_{R^N}\frac{|u|^{2^*(s)}}{|x|^s}\mathrm{d}x\right)^{\frac{2}{2^*(s)}}\leqslant C(s)\int_{R^N}|\Delta u|^2\mathrm{d}x,\quad\forall u\in D^{2,2}(R^N).\tag{3}$$

由(2)和(3)式我们定义下面的 Rellich-Sobolev 常数[1]:

收稿日期　2017-12-08

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基金项目　国家自然科学基金资助项目(11601530);中南民族大学研究生科研创新基金资助项目(2018sycxjj111)

$$S(\mu):=\inf_{u\in D^{2,2}(R^N)\setminus\{0\}}\frac{\int_{R^N}\left(|\Delta u|^2-\mu\frac{u^2}{|x|^4}\right)\mathrm{d}x}{\left(\int_{R^N}\frac{u^{2^*(s)}}{|x|^s}\mathrm{d}x\right)^{\frac{2}{2^*(s)}}},$$

$\mu<\bar{\mu},0\leqslant s<4.$

假设(H_1)成立,由文[1]可知$S(\mu)$的达到函数是:

$$V_\mu^\varepsilon(x):=\varepsilon^{\frac{4-N}{2}}U_\mu(\varepsilon^{-1}x),\varepsilon>0,\tag{4}$$

其中$U_\mu(x)>0$是径向对称、单调递减的,并且满足下面条件[1,2]:

$$\int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2-\mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right)\mathrm{d}x=\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}\mathrm{d}x=S(\mu)^{\frac{N-s}{4-s}}.$$

从文[1]可知$U_\mu(x)$是下列方程组的解:

$$\begin{cases}\Delta^2u-\mu\frac{u}{|x|^4}=\frac{u^{2^*(s)-1}}{|x|^s},x\in R^N\setminus\{0\},\\ u>0.\end{cases}$$

下面在$D^{2,2}(R^N)\setminus\{0\}$中定义最佳常数[1,3]:

$S_{\eta,\alpha,\beta}(\mu):=$

$$\inf_{(u,v)\in D}\frac{\int_{R^N}\left(|\Delta u|^2+|\Delta v|^2-\mu\frac{u^2}{|x|^4}-\mu\frac{v^2}{|x|^4}\right)\mathrm{d}x}{\left(\int_{R^N}\frac{\left(|u|^{2^*(s)}+\eta|u|^\alpha|v|^\beta+|v|^{2^*(s)}\right)}{|x|^s}\mathrm{d}x\right)^{\frac{2}{2^*(s)}}}.\tag{5}$$

我们设:

$$f(\tau):=\frac{1+\tau^2}{(1+\eta\tau^\beta+\tau^{2^*(s)})^{\frac{2}{2^*(s)}}},\tau\geqslant0,\tag{6}$$

$$f(\tau_{\min}):=\min_{\tau\geqslant0}f(\tau)>0,$$

这里$\tau_{\min}\geqslant0$是$f(\tau)$的最小值点.

本文中的基态解是指在所有解中对应能量值最小的解,因此(5)式被达到时的解是基态解.

其次,我们证明下列方程组解的存在性:

$$\begin{cases}\Delta^2u-\mu\frac{u}{|x|^4}=\frac{|u|^{2^*(s)-2}u}{|x|^s}+\frac{\eta\alpha}{2^*(s)}\frac{|u|^{\alpha-2}|v|^\beta u}{|x|^s}+\\ \quad a_1u+a_2v,x\in\Omega,\\ \Delta^2v-\mu\frac{v}{|x|^4}=\frac{|v|^{2^*(s)-2}v}{|x|^s}+\frac{\eta\beta}{2^*(s)}\frac{|u|^\alpha|v|^{\beta-2}v}{|x|^s}+\\ \quad a_2u+a_3v,x\in\Omega,\\ u,v\in H_0^2(\Omega),\end{cases}\tag{7}$$

其中$\Omega\subset R^N$是包含原点的有界光滑区域,$a_i\in R^N$,$i=1,2,3.H_0^2(\Omega)$表示$C_0^\infty(\Omega)$关于$\left(\int_{R^N}|\Delta\cdot|\,\mathrm{d}x\right)^{\frac{1}{2}}$的完备化空间.双调和方程的解的存在性已经被很多学者研究过[1,2,4],因此我们可以借鉴他们的方法,在$H:=H_0^2(\Omega)$中定义:

$$\|u\|^2:=\int_\Omega\left(|\Delta u|^2-\mu\frac{u^2}{|x|^4}\right)\mathrm{d}x.$$

方程组(7)对应的能量泛函为:

$J(u,v):=$

$$\frac{1}{2}\int_\Omega\left(|\Delta u|^2+|\Delta v|^2-\mu\frac{u^2}{|x|^4}-\mu\frac{v^2}{|x|^4}\right)\mathrm{d}x-\frac{1}{2^*(s)}\int_\Omega\left(\frac{|u|^{2^*(s)}}{|x|^s}+\frac{|v|^{2^*(s)}}{|x|^s}+\eta\frac{|u|^\alpha|v|^\beta}{|x|^s}\right)\mathrm{d}x-\frac{1}{2}\int_\Omega(a_1u^2+2a_2uv+a_3v^2)\mathrm{d}x,\tag{8}$$

其中$u,v\in H,J\in C^1(H\times H,R)$.对于$(u,v)\in H\times H\setminus(0,0)$,如果$\langle J'(u,v),(\varphi,\phi)\rangle=0,\forall(\varphi,\phi)\in H\times H$,那么我们称$(u,v)$是方程组(7)式的解.接下来,我们假设:

(H_2) $a_i>0,i=1,2,3,a_1a_3-a_2^2>0,0<\lambda_1\leqslant\lambda_2<\hat{\lambda}$.

其中λ_1和λ_2是矩阵$A:=\begin{pmatrix}a_1&a_2\\a_2&a_3\end{pmatrix}$的特征值,$\hat{\lambda}$是下列方程组的第一特征值:

$$\begin{cases}\Delta^2u-\mu\frac{u}{|x|^4}=\lambda u,x\in\Omega,\\ u=\frac{\partial u}{\partial n}=0,x\in\partial\Omega.\end{cases}$$

下面定义本文中一些常用的符号:

$$u^*:=\frac{1}{16}(N^2-16)(N^2-8N),N\geqslant9.$$

$$\delta:=\frac{N-4}{2},\delta(s):=\frac{2(N-4)}{4-s},0\leqslant s<4.$$

$$\varphi(\mu):=1-\frac{\sqrt{N^2-4N+8-4\sqrt{(N-2)^2+\mu}}}{N-4},\mu\in[0,\bar{\mu}].$$

$$\alpha(\mu):=\delta\varphi(\mu),b(\mu):=\delta(2-\varphi(\mu)),\mu\in[0,\bar{\mu}].$$

本文的结论如下:

定理1 设(u_0,v_0)是方程组(1)的基态解,若$\varepsilon>0,0\leqslant s<3$,并且条件$(\mathrm{H}_1)$成立,则$(u_0,v_0)=(rV_\mu^\varepsilon(x),tV_\mu^\varepsilon(x)),t=r\tau_{\min}$,其中$r$满足:$r^{2^*(s)-2}=\frac{2^*(s)}{2^*(s)+\eta\alpha\tau_{\min}^\beta},r>0.$

定理 2　假设 $N > 8, 0 \leqslant \mu \leqslant \mu^*$,并且$(H_1)$和$(H_2)$成立,则方程组(7)至少存在一个解$(u,v) \in (H\backslash\{0\})^2$.

为了书写方便,全文用 C 表示常数;用 $O(\varepsilon^t)$ 表示 $|O(\varepsilon^t)|/\varepsilon^t \leqslant C$;用 $o(\varepsilon^t)$ 表示 $|o(\varepsilon^t)|/\varepsilon^t \to 0$;用 $O_1(\varepsilon^t)$ 表示 $C_1\varepsilon^t \leqslant |O_1(\varepsilon^t)| \leqslant C_2\varepsilon^t$(当 ε 足够小时),并且省略 dx.

2　定理 1 的证明

引理 1　假设条件(H_1)成立,$V_\mu^\varepsilon(x)$ 是 $S(\mu)$ 的达到函数,则 $S_{\eta,\alpha,\beta}(\mu) = f(\tau_{\min})S(\mu)$ 并且 $S_{\eta,\alpha,\beta}(\mu)$ 的达到函数对是$(V_\mu^\varepsilon(x), \tau_{\min}V_\mu^\varepsilon(x)), \forall \varepsilon > 0$.

证明　与文献[3]中定理 1.1 及文献[5]中定理 5 类似,在此省略.

由 $f'(\tau_{\min}) = 0$ 可以推出 $\tau_{\min}$ 是下列方程的一个根:

$$g(\tau) := 2^*(s) + \eta\alpha\tau^\beta - \eta\beta\tau^{\beta-2} - 2^*(s)\tau^{2^*(s)-2} = 0, \quad \tau > 0. \tag{9}$$

假设 $\varepsilon > 0, u \in D^{2,2}(R^N), r > 0$,若方程组(1)有正解并且解的形式为$(ru, tu)$,则可得到:

$$(2^*(s) + \eta\alpha\tau^\beta)r^{2^*(s)-2} = 2^*(s) = (2^*(s)\tau^{2^*(s)-2} + \eta\beta\tau^{\beta-2})r^{2^*(s)-2}, t = \tau r, u = V_\mu^\varepsilon. \tag{10}$$

引理 2　假设 $N \geqslant 5, 0 \leqslant s < 3, 0 < \eta < +\infty, \alpha, \beta > 1, \alpha + \beta = 2^*(s)$,那么方程(9)至少有一个根 $\tau > 0$,且$(rV_\mu^\varepsilon(x), tV_\mu^\varepsilon(x))$是(1)式的解,其中 t 和 r 满足:

$$t = \tau r, r^{2^*(s)-2} = \frac{2^*(s)}{2^*(s) + \eta\alpha\tau^\beta}.$$

证明　与文献[5]中引理 2.2 类似,在此省略.

定理 1 的证明　我们应用与文献[5]中定理 1.1 类似的方法来证明本定理.设(u_0, v_0)是方程组(1)的基态解,首先我们证明:

$$\int_{R^N} \frac{|u_0|^{2^*(s)}}{|x|^s} = r^{2^*(s)} \int_{R^N} \frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}. \tag{11}$$

考虑下列带有参数 $\tilde{\mu} > 0$ 的方程组:

$$\begin{cases} \Delta^2 u - \mu \dfrac{u}{|x|^4} = \tilde{\mu} \dfrac{u^{2^*(s)-1}}{|x|^s} + \dfrac{\eta\alpha}{2^*(s)} \dfrac{u^{\alpha-1}v^\beta}{|x|^s}, \\ \Delta^2 v - \mu \dfrac{v}{|x|^4} = \dfrac{v^{2^*(s)-1}}{|x|^s} + \dfrac{\eta\beta}{2^*(s)} \dfrac{u^\alpha v^{\beta-1}}{|x|^s}, \\ u, v > 0, x \in R^N\backslash\{0\}. \end{cases}$$

同样地,记:

$$f_1(\tau) := \frac{1 + \tau^2}{(\tilde{\mu} + \eta\tau^\beta + \tau^{2^*(s)})^{\frac{2}{2^*(s)}}}. \tag{12}$$

$$f_1(\tau_{\min}^*) := \min_{\tau \geqslant 0} f_1(\tau).$$

那么 $\tau_{\min}^*(\tilde{\mu})$ 满足:$2^*(s)\tilde{\mu} + \eta\alpha\tau^\beta - \eta\beta\tau^{\beta-2} - 2^*(s)\tau^{2^*(s)-2} = 0$.

接下来,我们定义函数 $F(\tilde{\mu}, \tau) := 2^*(s)\tilde{\mu} + \eta\alpha\tau^\beta - \eta\beta\tau^{\beta-2} - 2^*(s)\tau^{2^*(s)-2}$,那么,$\dfrac{\partial F}{\partial \tau} = \tau^{\beta-3}(\eta\alpha\beta\tau^2 - \eta\beta(\beta-2) - 2^*(s)(2^*(s)-2)\tau^\alpha)$.因为 $\tau_{\min}$ 是 $f(\tau)$ 的最小值,所以 $g(\tau_{\min}) = 0, g'(\tau_{\min}) > 0$,则 $F(1, \tau_{\min}) = 0, \dfrac{\partial F}{\partial \tau}(1, \tau_{\min}) > 0$.由隐函数定理可得到,对于 $0 < \varepsilon < 1, \tilde{\mu} \in (1-\varepsilon, 1+\varepsilon), \tau_{\min}^*(\tilde{\mu}), r^*(\tilde{\mu}), t^*(\tilde{\mu}) \in C^1$.此外,根据文献[7]定义基态能为:

$$B(\tilde{\mu}) := (r^*(\tilde{\mu}))^2(1 + (\tau_{\min}^*(\tilde{\mu}))^2)B_1 \in C^1((1-\varepsilon, 1+\varepsilon), R), B_1 := \frac{4-s}{2(N-s)} \int_{R^N} \frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}.$$

容易得到:

$$B(\tilde{\mu}) = \inf_{(u,v) \in D} \max_{t>0} E(tu, tv). \tag{13}$$

$$E := \frac{1}{2}\int_{R^N}\left(|\Delta u|^2 + |\Delta v|^2 - \mu\frac{u^2}{|x|^4} - \mu\frac{v^2}{|x|^4}\right) - \frac{1}{2^*(s)}\int_{R^N}\left(\tilde{\mu}\frac{|u|^{2^*(s)}}{|x|^s} + \frac{|v|^{2^*(s)}}{|x|^s} + \eta\frac{|u|^\alpha|v|^\beta}{|x|^s}\right),$$

令

$$I := \int_{R^N}\left(|\Delta u_0|^2 + |\Delta v_0|^2 - \mu\frac{|u_0|^2}{|x|^4} - \mu\frac{|v_0|^2}{|x|^4}\right),$$

$$D := \int_{R^N}\left(\frac{|v_0|^{2^*(s)}}{|x|^s} + \eta\frac{|u_0|^\alpha|v_0|^\beta}{|x|^s}\right),$$

$$G := \int_{R^N}\frac{|u_0|^{2^*(s)}}{|x|^s},$$

存在 $t(\tilde{\mu}) > 0$,使得 $\max_{t>0} E(tu_0, tv_0) = E(t(\tilde{\mu})u_0, t(\tilde{\mu})v_0)$,并且满足条件:

$$K(\tilde{\mu}, t(\tilde{\mu})) = 0, K(\tilde{\mu}, t) := t^{2^*(s)-2}(\tilde{\mu}G + D) - I.$$

需要指出 $K(1,1) = 0, \dfrac{\partial K}{\partial t}(1,1) > 0$ 且 $K(\tilde{\mu}, t(\tilde{\mu})) \equiv 0$.由隐函数定理得,存在 $0 < \varepsilon_1 \leqslant \varepsilon$,使得 $t(\tilde{\mu}) \in C^1((1-\varepsilon, 1+\varepsilon_1), R)$ 且 $t'(1) = -\dfrac{G}{(2^*(s)-2)(G+D)}$.对 $t(\tilde{\mu})$ 进行泰勒展开,可以得到 $t(\tilde{\mu}) = 1 + t'(1)(\tilde{\mu}-1) + O((\tilde{\mu}-1)^2), t^2(\tilde{\mu}) = 1 + 2t'(1)(\tilde{\mu}-1) + O((\tilde{\mu}-1)^2)$.由 $K(1,1) = 0$ 得到 $I = G + D = \dfrac{2(N-s)}{4-s}B(1)$,那么由(13)式,我

们有:

$$B(\tilde{\mu}) \leqslant B(1) - \frac{G}{2^*(s)}(\tilde{\mu}-1) + O((\tilde{\mu}-1)^2).$$

所以得到,当 $\tilde{\mu}\uparrow 0$ 时,$\frac{B(\tilde{\mu})-B(1)}{\tilde{\mu}-1} \geqslant -\frac{G}{2^*(s)} + O(\tilde{\mu}-1)$,则 $B'(1) \geqslant -\frac{G}{2^*(s)}$.同样的我们得到,当 $\tilde{\mu}\downarrow 1$ 时,$\frac{B(\tilde{\mu})-B(1)}{\tilde{\mu}-1} \leqslant -\frac{G}{2^*(s)} + O(\tilde{\mu}-1)$,则 $B'(1) \leqslant -\frac{G}{2^*(s)}$.因此 $B'(1) = -\frac{G}{2^*(s)} = -\frac{1}{2^*(s)}\cdot\int_{R^N}\frac{|u_0|^{2^*(s)}}{|x|^s}$.此外,因为 $(rV_\mu^\varepsilon(x), tV_\mu^\varepsilon(x))$ 是方程组(1)的基态解,其中 t 和 r 满足:$t = r\tau_{\min}$,$r^{2^*(s)-2} = \frac{2^*(s)}{2^*(s)+\eta\alpha\tau_{\min}^\beta}$,所以:

$$B'(1) = -\frac{r^{2^*(s)}}{2^*(s)}\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}.$$

因此(11)式成立.

同理可得:

$$\begin{cases}\int_{R^N}\frac{|v_0|^{2^*(s)}}{|x|^s} = t^{2^*(s)}\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s},\\ \int_{R^N}\frac{|u_0|^\alpha|v_0|^\beta}{|x|^s} = r^\alpha t^\beta\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}.\end{cases} \tag{14}$$

因此由(11)、(14)式得:

$$\begin{cases}\int_{R^N}\frac{|u_0|^\alpha|v_0|^\beta}{|x|^s} = r^{\alpha-2^*(s)}t^\beta\int_{R^N}\frac{|u_0|^{2^*(s)}}{|x|^s},\\ \int_{R^N}\frac{|u_0|^\alpha|v_0|^\beta}{|x|^s} = r^\alpha t^{\beta-2^*(s)}\int_{R^N}\frac{|v_0|^{2^*(s)}}{|x|^s}.\end{cases} \tag{15}$$

由(10)式得到:

$$\left(r^{2^*(s)-2} + \frac{1}{2^*(s)}\eta\alpha t^\beta r^{\alpha-2}\right) = 1 = \left(t^{2^*(s)-2} + \frac{1}{2^*(s)}\eta\beta t^{\beta-2}r^\alpha\right). \tag{16}$$

定义 $(u_1,v_1) := \left(\frac{u_0}{r},\frac{v_0}{r}\right)$,由(15)、(16)式我们得到:

$$\int_{R^N}\left(|\Delta u_1|^2 - \mu\frac{|u_1|^2}{|x|^4}\right) = \int_{R^N}\frac{|u_1|^{2^*(s)}}{|x|^s},$$

同理可得 $\int_{R^N}\left(|\Delta v_1|^2 - \mu\frac{|v_1|^2}{|x|^4}\right) = \int_{R^N}\frac{|v_1|^{2^*(s)}}{|x|^s}$.

因此,

$$\int_{R^N}\left(|\Delta u_1|^2 - \mu\frac{|u_1|^2}{|x|^4}\right) \geqslant \int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2 - \mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right),$$

$$\int_{R^N}\left(|\Delta v_1|^2 - \mu\frac{|v_1|^2}{|x|^4}\right) \geqslant \int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2 - \mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right).$$

因为 (u_0,v_0) 及 $(rV_\mu^\varepsilon(x), tV_\mu^\varepsilon(x))$ 是方程组(1)的基态解,所以我们得到:

$$\frac{4-s}{2(N-s)}(r^2+t^2)\int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2 - \mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right) =$$

$$\frac{4-s}{2(N-s)}\int_{R^N}\left(|\Delta u_0|^2 - \mu\frac{|u_0|^2}{|x|^4} + |\Delta v_0|^2 - \mu\frac{|v_0|^2}{|x|^4}\right) = \frac{4-s}{2(N-s)}\left(r^2\int_{R^N}\left(|\Delta u_1|^2 - \mu\frac{|u_1|^2}{|x|^4}\right) + t^2\int_{R^N}\left(|\Delta v_1|^2 - \mu\frac{|v_1|^2}{|x|^4}\right)\right) \geqslant \frac{4-s}{2(N-s)}(r^2+t^2)\int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2 - \mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right).$$

由此

$$\int_{R^N}\left(|\Delta u_1|^2 - \mu\frac{|u_1|^2}{|x|^4}\right) = \int_{R^N}\left(|\Delta V_\mu^\varepsilon(x)|^2 - \mu\frac{|V_\mu^\varepsilon(x)|^2}{|x|^4}\right) = \int_{R^N}\left(|\Delta v_1|^2 - \mu\frac{|v_1|^2}{|x|^4}\right). \tag{17}$$

所以 u_1, v_1 是下面方程的基态解:

$$\Delta^2 - \mu\frac{u}{|x|^4} = \frac{u^{2^*(s)-1}}{|x|^s}u, u \in D^{2,2}(R^N).$$

由 Hölder 不等式、(11)式及(14)式得到:

$$\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s} = \int_{R^N}\frac{|u_1|^\alpha|v_1|^\beta}{|x|^s} \leqslant \frac{\alpha}{2^*(s)}\int_{R^N}\frac{|u_1|^{2^*(s)}}{|x|^s} + \frac{\beta}{2^*(s)}\int_{R^N}\frac{|v_1|^{2^*(s)}}{|x|^s} = \frac{\alpha}{2^*(s)}\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s} + \frac{\beta}{2^*(s)}\int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s} = \int_{R^N}\frac{|V_\mu^\varepsilon(x)|^{2^*(s)}}{|x|^s}.$$

因此,上式的不等式可以换成等式.且由上式和(17)式我们得到 $u_1 = v_1$,所以由唯一性及(15)式我们得

到对 $\varepsilon > 0$，有 $u_1 = v_1 = V_\mu^\varepsilon(x)$，所以 $(u_0, v_0) = (rV_\mu^\varepsilon(x), tV_\mu^\varepsilon(x))$. 定理 1 证毕.

3　定理 2 的相关引理

引理 3　假设 (H_1) 和 (H_2) 成立，则对任意 $c < c^* := \frac{4-s}{2(N-s)}(S_{\eta,\alpha,\beta}(\mu))^{\frac{N-s}{4-s}}$，方程组(7)的能量泛函 J 满足 $(PS)_c$ 条件.

证明　与文献[3]中引理 2.4.1 类似，在此省略.

引理 4[1]　假设 $N \geqslant 5, 0 \leqslant s < 4, 0 \leqslant \mu < \bar{\mu}$，当 $\varepsilon \to 0$ 时，下列估计成立：

$$\int_\Omega \left(|\Delta u_\varepsilon|^2 - \mu \frac{|u_\varepsilon|^2}{|x|^4}\right) = S(\mu)^{\frac{N-s}{4-s}} + O(\varepsilon^{2(b(\mu)-\delta)}),$$

$$\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} = S(\mu)^{\frac{N-s}{4-s}} + O(\varepsilon^{2^*(s)(b(\mu)-\delta)}).$$

当 $N \geqslant 8$，

$$\int_\Omega |u_\varepsilon|^2 = \begin{cases} O_1(\varepsilon^4), 0 \leqslant \mu < \mu^*, \\ O_1(\varepsilon^4 |\ln\varepsilon|), \mu = \mu^*. \end{cases}$$

引理 5　若 $N > 8$，(H_1) 和 (H_2) 成立，并且 $0 \leqslant \mu \leqslant \mu^*$，那么 $\sup_{t>0} J(t\tilde{\mu}, t\tilde{v}) < c^*$.

证明　由假设条件 (H_2)，矩阵 $Q(u,v) := (u, v)A\begin{pmatrix} u \\ v \end{pmatrix} = a_1u^2 + 2a_2uv + a_3v^2$ 是正定矩阵，并且满足：$\lambda_1(u^2+v^2) \leqslant Q(u,v) \leqslant \lambda_2(u^2+v^2), \forall u,v \in H$. 定义下面函数：

$$g_1(t) := J(tu_\varepsilon, t\tau_{\min}u_\varepsilon) \leqslant \frac{t^2}{2}(1+\tau_{\min}^2)\int_\Omega\left(|\Delta u_\varepsilon|^2 - \mu\frac{|u_\varepsilon|^2}{|x|^s} - \lambda_1 u_\varepsilon^2\right) - \frac{t^{2^*(s)}}{2^*(s)}(1+\eta\tau_{\min}^\beta + \tau_{\min}^{2^*(s)})\int_\Omega\left(\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\right) = \bar{g}_1(t) - \frac{t^2}{2}(1+\tau_{\min}^2)\lambda_1\int_{R^N}|u_\varepsilon|^2.$$

另一方面有：

$$\max_{t>0}\left(\frac{t^2}{2}B' - \frac{t^{2^*(s)}}{2^*(s)}B''\right) = \frac{4-s}{2(N-s)}(B')^{\frac{N-s}{4-s}}(B'')^{\frac{N-4}{s-4}}, B', B'' > 0,$$

注意到：

$$0 \leqslant \mu < u^* \Leftrightarrow b(\mu) - \delta > 2, \mu = \mu^* \Leftrightarrow b(\mu) - \delta = 2, 2^*(s) > 2. \tag{18}$$

由(18)式、引理 1 和引理 4 得出：

$$\max_{t>0}\bar{g}_1(t) \leqslant \frac{4-s}{2(N-s)}(f(\tau_{\min}))^{\frac{N-s}{4-s}} \cdot \left(\int_\Omega\left(|\Delta u_\varepsilon|^2 - \mu\frac{|u_\varepsilon|^2}{|x|^4}\right)\right)^{\frac{N-s}{4-s}}\left(\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\right)^{\frac{N-4}{s-4}} \leqslant \frac{4-s}{2(N-s)}(S_{\eta,\alpha,\beta}(\mu))^{\frac{N-s}{4-s}} + O(\varepsilon^{2(b(\mu)-\delta)}).$$

由上式及引理 4 得到：

$$g_1(t) \leqslant \frac{4-s}{2(N-s)}(S_{\eta,\alpha,\beta}(\mu))^{\frac{N-s}{4-s}} + O(\varepsilon^{2(b(\mu)-\delta)}) - \frac{t^2}{2}(1+\tau_{\min}^2)\int_\Omega \lambda_1|u_\varepsilon|^2 \leqslant c^* + O(\varepsilon^{2(b(\mu)-\delta)}) - C\int_\Omega|u_\varepsilon|^2 < c^*.$$

引理 5 证毕.

定理 2 的证明过程与文献[3]中定理 1.2 类似，在此省略.

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