基于 SV 形态学方法的图像分解问题的研究

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摘　要　在固定结构元的形态学方法对图像分解问题研究的基础上,进一步研究了基于空间变化的形态学方法的图像内、外骨架提取及图像分解问题,建立了图像空间变化的内、外骨架变换,并将内、外骨架同时作为分解成分建立了图像分解公式的推广形式,所有公式的建立都给出了相应的推算和证明过程.

关键词　空间变化的结构元;内骨架;外骨架;骨架变换
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Research of Image Decomposition Based on Spatially-Variant Morphology Method

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Abstract　Based on the research of image decomposition by the morphology method of fixed structuring element, the problem of internal skeleton and external skeleton extraction and image decomposition by the spatially-variant morphology method is further studied. Internal and external skeleton transform of image spatially-variant is established, and the internal and external skeleton are simultaneously used as decomposition components to establish the generalized form of image decomposition formula. The corresponding calculation and proof process of all formulas are provided.

Keywords　spatially-variant structuring element; internal skeleton; external skeleton; skeleton transform

最初的数学形态学方法主要采用固定结构元建立相关的形态学理论,但人们很快发现单一的固定结构元在实际应用中存在许多缺陷,由此结构元的自适应性研究成为改进这一问题的一个方向. 以 Serra 为代表的学者们[1-3]早在形态学建立初期就提出了结构函数的思想,将结构元视为空间位置变化的函数,Charif-Chefchaouni M 和 Schonfeld D 等人在文献[4,5]中将其命名为空间变化(Spatially-Variant,简写为 SV)的结构元(简称 SV 结构元),并将固定结构元的形态学方法推广到 SV 结构元的情形[6],对具有 SV 结构元的形态学(简称 SV 形态学,即 Spatially-Variant Morphology)方法进行了一系列的研究,使得基于固定结构元的形态学理论得以推广.

图像分解与重构方法的研究是图像分析中的一个热点问题,有大量成熟的研究结果[7-9]. Xu J 对基于骨架的图像分解及重构问题有一系列的研究成果[10,11],其主要采用经典的数学形态学方法进行内、外骨架的提取,利用固定结构元的形态变换建立相关的图像分解及重构公式. Xu J 在文献[12]中,将基于固定结构元的内、外骨架变换交替运用于图像的分解,得出了更为一般的分解算法,其优势在于它能够通过较少的骨架点构造出质量较高的近似图像. SV 形态学方法的建立,使得对结构元的选取更具自适应性,结构元也从具体的形态变为更为抽象的空间变化的形式,这使得 SV 形态变换无论在理论上还是在实际应用中都更具优势,SV 结构元的适当选取能更有效地实现对目标对象的相关处理. 鉴于 SV 形态学方法的这一特点,本文在 Xu J 关于骨架变换研究的基础上,将 SV 形态学方法应用于建立更具一般性的 SV 骨架变换,应用 SV 结构元及 SV 形态变换实现对目标对象基于骨架的分解和重

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构，最终获得的结果推广了已有文献的相关结论.

1 基础知识

关于SV形态变换，Bouaynaya N和Schonfeld D有较为系统的研究[5,6]，其中最为重要的内容之一，是将SV形态学中的SV结构元定义为一个由点到集合的映射$\theta: E \to P(E)$，其中$P(E)$为幂集，且SV结构元映射θ依继承序$\theta_1 \leqslant \theta_2 \Leftrightarrow \theta_1(z) \subseteq \theta_2(z)$构成偏序集. SV结构元$\theta$的转置结构元定义为：$\theta'(y) = \{z \in E: y \in \theta(z), y \in E\}$. 对于任意的$X \in P(E)$，4个最基本的SV形态腐蚀、膨胀、开、闭变换分别定义为：

$$\varepsilon_\theta(X) = \{z \in E \mid \theta(z) \subseteq X\} = \bigcap_{x \in X^c} \theta'^c(x), \tag{1}$$

$$\sigma_\theta(X) = \{z \in E \mid \theta'(z) \cap X \neq \varnothing\} = \bigcup_{x \in X} \theta(x), \tag{2}$$

$$\gamma_\theta(X) = \sigma_\theta(\varepsilon_\theta(X)) = \cup\{(\theta(y) \mid \theta(y) \subseteq X; y \in E\}, \tag{3}$$

$$\phi_\theta(X) = \varepsilon_\theta(\sigma_\theta(X)) = \{z \in E \mid \theta(y) \cap X \neq \varnothing, \forall \theta(y) \mid z \in \theta(y)\}. \tag{4}$$

在以上定义的基础上，首先给出4个引理，作为后续工作开展的基础.

引理1 对于$A, B \in P(E)$，有$\sigma_\theta(A \cup B) = \sigma_\theta(A) \cup \sigma_\theta(B)$.

引理2 对于$A, B \in P(E)$，有$\varepsilon_\theta(A \backslash B) = \varepsilon_\theta(A) \backslash \sigma_{\theta'}(B)$.

引理3 对于SV结构元θ_1和θ_2，有$\sigma_{\theta_2}(\sigma_{\theta_1}) = \sigma_{\sigma_{\theta_2}(\theta_1)}$，即$\sigma_{\theta_2}(\sigma_{\theta_1})(X) = \sigma_{\sigma_{\theta_2}(\theta_1)}(X)$.

引理4 对于SV结构元θ_1和θ_2，有$\varepsilon_{\theta_2}(\varepsilon_{\theta_1}) = \varepsilon_{\sigma_{\theta_1}(\theta_2)}$，即$\varepsilon_{\theta_2}(\varepsilon_{\theta_1})(X) = \varepsilon_{\sigma_{\theta_1}(\theta_2)}(X)$.

2 SV的内骨架变换

根据引理3，以θ为结构元作二次SV膨胀σ_θ，得到以$\sigma_\theta(\theta)$为结构元的膨胀运算，若令$\sigma_\theta(\theta) = \theta_2$，则有$\sigma_\theta(\sigma_\theta) = \sigma_{\theta_2}$；同理，$\sigma_\theta(\sigma_\theta(\sigma_\theta)) = \sigma_{\sigma_{\sigma_\theta(\theta)}(\theta)}$，相当于以$\theta$为结构元作三次膨胀，得到以$\sigma_{\sigma_\theta(\theta)}(\theta) = \theta_3$为结构元的膨胀运算$\sigma_\theta(\sigma_\theta(\sigma_\theta)) = \sigma_{\theta_3}$，依此类推$\underbrace{\sigma_\theta(\cdots(\sigma_\theta(\sigma_\theta)))}_{n}$相当于以$\theta$为结构元作$n$次膨胀，得到以$\underbrace{\sigma_\theta(\cdots(\sigma_\theta(\sigma_\theta)))}_{n-1}(\theta) = \theta_n$为结构元的膨胀运算$\underbrace{\sigma_\theta(\cdots(\sigma_\theta(\sigma_\theta)))}_{n} = \sigma_{\theta_n}$. 结合引理4，亦有$\underbrace{\varepsilon_\theta(\cdots(\varepsilon_\theta(\varepsilon_\theta)))}_{n} = \varepsilon_{\theta_n}$，特别地，当$n = 1$时，$\sigma_{\theta_n}$和$\varepsilon_{\theta_n}$即是$\sigma_\theta$和$\varepsilon_\theta$. 关于$\theta'$类似地有：

$\underbrace{\sigma_{\theta'}(\cdots(\sigma_{\theta'}(\sigma_{\theta'})))}_{n} = \sigma_{\theta'_n}$，$\underbrace{\varepsilon_{\theta'}(\cdots(\varepsilon_{\theta'}(\varepsilon_{\theta'})))}_{n} = \varepsilon_{\theta'_n}$，

同样，当$n = 1$时，$\sigma_{\theta'_n}$和$\varepsilon_{\theta'_n}$即是$\sigma_{\theta'}$和$\varepsilon_{\theta'}$.

利用SV开运算的非扩展性[5]，即$\gamma_\theta(X) \subseteq X$，对$X$实施一次开运算$\gamma_\theta(X)$，得：

$$X = \gamma_\theta(X) \cup S_0 = \sigma_\theta(\varepsilon_\theta(X)) \cup S_0 = \sigma_\theta(X_1) \cup S_0, \tag{5}$$

其中$X_1 = \varepsilon_\theta(X)$，$S_0 = X \backslash \gamma_\theta(X)$. 对$X_1$作开运算$\gamma_\theta(X_1)$，将$X_1$表示成如下形式：

$$X_1 = \gamma_\theta(X_1) \cup S_1 = \sigma_\theta(\varepsilon_\theta(X_1)) \cup S_1 = \sigma_\theta(X_2) \cup S_1, \tag{6}$$

其中$X_2 = \varepsilon_\theta(X_1) = \varepsilon_\theta(\varepsilon_\theta(X)) = \varepsilon_{\theta_2}(X)$，$S_1 = X_1 \backslash \gamma_\theta(X_1)$，将(6)式代入到(5)式，并利用引理1可得：

$$\begin{aligned} X &= \sigma_\theta(X_1) \cup S_0 = \sigma_\theta(\sigma_\theta(X_2) \cup S_1) \cup S_0 = \\ &\sigma_{\theta_2}(X_2) \cup \sigma_\theta(S_1) \cup S_0 = \\ &\sigma_{\theta_2}(\varepsilon_{\theta_2}(X)) \cup \sigma_\theta(S_1) \cup S_0 = \\ &\gamma_{\theta_2}(X) \cup \sigma_\theta(S_1) \cup S_0. \end{aligned} \tag{7}$$

再对X_2进行一次开运算$\gamma_\theta(X_2)$，将X_2表示成如下形式：

$$X_2 = \gamma_\theta(X_2) \cup S_2 = \sigma_\theta(\varepsilon_\theta(X_2)) \cup S_2 = \sigma_\theta(X_3) \cup S_2, \tag{8}$$

其中$X_3 = \varepsilon_\theta(X_2) = \varepsilon_\theta(\varepsilon_\theta(\varepsilon_\theta(X))) = \varepsilon_{\theta_3}(X)$，$S_2 = X_2 \backslash \gamma_\theta(X_2)$，将(8)式代入(7)式，并利用引理1可得：

$$\begin{aligned} X &= \sigma_{\theta_2}(\sigma_\theta(X_3) \cup S_2) \cup \sigma_\theta(S_1) \cup S_0 = \\ &\sigma_{\theta_3}(X_3) \cup \sigma_{\theta_2}(S_2) \cup \sigma_\theta(S_1) \cup S_0 = \\ &\sigma_{\theta_3}(\varepsilon_{\theta_3}(X)) \cup \sigma_{\theta_2}(S_2) \cup \sigma_\theta(S_1) \cup S_0 = \\ &\gamma_{\theta_3}(X) \cup \sigma_{\theta_2}(S_2) \cup \sigma_\theta(S_1) \cup S_0. \end{aligned} \tag{9}$$

重复以上过程，最后可以得到如下公式：

$$\begin{aligned} X &= \sigma_{\theta_n}(X_n) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \cup \sigma_{\theta_{n-2}}(S_{n-2}) \cup \cdots \cup \sigma_\theta(S_1) \cup S_0 = \sigma_{\theta_n}(\varepsilon_{\theta_n}(X)) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \cup \\ &\sigma_{\theta_{n-2}}(S_{n-2}) \cup \cdots \cup \sigma_\theta(S_1) \cup S_0 = \gamma_{\theta_n}(X) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \cup \sigma_{\theta_{n-2}}(S_{n-2}) \cup \cdots \cup \sigma_\theta(S_1) \cup S_0, \end{aligned} \tag{10}$$

其中$X_n = \varepsilon_\theta(X_{n-1}) = \underbrace{\varepsilon_\theta(\cdots(\varepsilon_\theta(\varepsilon_\theta)))}_{n}(X) = \varepsilon_{\theta_n}(X)$，$S_{n-1} = X_{n-1} \backslash \gamma_\theta(X_{n-1})$，将(10)式称为图像的SV内骨架变换公式，$S_i(i = 0, 1, 2, \cdots, n-1)$称为图像的$i$级内骨架.

3　SV 的外骨架变换

关于外骨架,文献[12]中有明确的阐述,外骨架作为一种不同于内骨架的图像特征,在实际应用中有其特有的作用.

由 SV 闭运算的扩展性[5],即 $X \subseteq \phi_{\theta'}(X)$,对 X 进行一次闭运算 $\phi_{\theta'}(X)$,有:

$$X = \phi_{\theta'}(X) \backslash T_0 = \varepsilon_{\theta'}(\sigma_{\theta'}(X)) \backslash T_0 = \varepsilon_{\theta'}(X_1) \backslash T_0, \tag{11}$$

其中 $X_1 = \sigma_{\theta'}(X)$,$T_0 = \phi_{\theta'}(X) \backslash X$,对 X_1 作类似的操作得:

$$X_1 = \phi_{\theta'}(X_1) \backslash T_1 = \varepsilon_{\theta'}(\sigma_{\theta'}(X_1)) \backslash T_1 = \varepsilon_{\theta'}(X_2) \backslash T_1, \tag{12}$$

其中 $X_2 = \sigma_{\theta'}(X_1) = \sigma_{\theta'}(\sigma_{\theta'}(X)) = \sigma_{\theta'_2}(X)$,$T_1 = \phi_{\theta'}(X_1) \backslash X_1$,将(12)式代入到(11)式,并利用引理 2 可得:

$$X = \varepsilon_{\theta'}(\varepsilon_{\theta'}(X_2) \backslash T_1) \backslash T_0 = \varepsilon_{\theta'_2}(X_2) \backslash \sigma_{\theta}(T_1) \backslash T_0 = \phi_{\theta'_2}(X) \backslash \sigma_{\theta}(T_1) \backslash T_0. \tag{13}$$

再对 X_2 进行一次闭运算 $\phi_{\theta'}(X_2)$,并将 X_2 表示成为:

$$X_2 = \phi_{\theta'}(X_2) \backslash T_2 = \varepsilon_{\theta'}(\sigma_{\theta'}(X_2)) \backslash T_2 = \varepsilon_{\theta'}(X_3) \backslash T_2, \tag{14}$$

其中 $X_3 = \sigma_{\theta'}(X_2) = \sigma_{\theta'_3}(X)$,$T_2 = \phi_{\theta'}(X_2) \backslash X_2$. 将(14)式代入到(13)式,并利用引理 2 可得:

$$\begin{aligned} X &= \varepsilon_{\theta'_2}(\varepsilon_{\theta'}(X_3) \backslash T_2) \backslash \sigma_{\theta}(T_1) \backslash T_0 = \\ &\varepsilon_{\theta'_3}(X_3) \backslash \sigma_{\theta_2}(T_2) \backslash \sigma_{\theta}(T_1) \backslash T_0 = \\ &\varepsilon_{\theta'_3}(\sigma_{\theta'_3}(X)) \backslash \sigma_{\theta_2}(T_2) \backslash \sigma_{\theta}(T_1) \backslash T_0 = \\ &\phi_{\theta'_3}(X) \backslash \sigma_{\theta_2}(T_2) \backslash \sigma_{\theta}(T_1) \backslash T_0. \end{aligned} \tag{15}$$

重复相同步骤,最后可以得到:

$$\begin{aligned} X &= \varepsilon_{\theta'_n}(X_n) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \cdots \backslash \sigma_{\theta}(T_1) \backslash T_0 = \\ &\varepsilon_{\theta'_n}(\sigma_{\theta'_n}(X_n)) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \cdots \backslash \sigma_{\theta}(T_1) \backslash T_0 = \\ &\phi_{\theta'_n}(X) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \cdots \backslash \sigma_{\theta}(T_1) \backslash T_0, \end{aligned} \tag{16}$$

其中 $X_n = \sigma_{\theta'}(X_{n-1}) = \underbrace{\sigma_{\theta'}(\cdots(\sigma_{\theta'}(\sigma_{\theta'})))}_{n}(X) = \sigma_{\theta'}(X)$,$T_{n-1} = \phi_{\theta'}(X_{n-1}) \backslash X_{n-1}$,将(16)式称为图像的 SV 外骨架变换公式,$T_i(i = 0,1,2,\cdots,n-1)$称为图像的 i 级外骨架.

4　内、外骨架统一的图像重构公式

以上分别利用内、外骨架给出图像的分解及重构公式,但单一运用内骨架或是外骨架进行分解往往产生大量的冗余,且骨架级数相对较高. 文献[12]通过实例说明综合运用内、外骨架能重构出高质量的近似图像,且用于重构的内、外骨架成分相对较少. 为此,以下将利用 SV 形态学方法,对文献[12]中的相关工作作进一步地推广.

4.1　近似公式的推导

在已建立的 SV 内、外骨架变换的基础上,利用得到的公式及结论,通过交替实施开、闭运算,建立基于内、外骨架的图像分解及重构公式.

首先对 X 进行一次开运算 $\gamma_{\theta}(X)$,将其表示为:

$$X = \gamma_{\theta}(X) \cup S_0 = X' \cup S_0, \tag{17}$$

其中 $X' = \gamma_{\theta}(X)$,$S_0 = X \backslash \gamma_{\theta}(X)$. 对 X' 进行一次闭运算 $\phi_{\theta'}(X')$,可得:

$$X' = \phi_{\theta'}(X') \backslash T_0 = X_1 \backslash T_0, \tag{18}$$

其中 $X_1 = \phi_{\theta'}(X')$,$T_0 = \phi_{\theta'}(X') \backslash X'$. 将(18)式代入到(17)式可得:

$$X = X_1 \backslash T_0 \cup S_0. \tag{19}$$

运用(7)式可以将 X_1 近似地表示为:

$$X_1 \approx \gamma_{\theta_2}(X_1) \cup \sigma_{\theta}(S_1) = X'_1 \cup \sigma_{\theta}(S_1) = \gamma_{\theta}(X_1), \tag{20}$$

其中 $X'_1 = \gamma_{\theta_2}(X_1)$,$S_1 = \varepsilon_{\theta}(X_1) \backslash \gamma_{\theta}(\varepsilon_{\theta}(X_1))$,将(20)式代入到(19)式中,得到关于 X 的第一个近似式:

$$X \approx X'_1 \cup \sigma_{\theta}(S_1) \backslash T_0 \cup S_0. \tag{21}$$

运用(13)式,将 X'_1 近似地表示为:

$$X'_1 \approx \phi_{\theta'_2}(X'_1) \backslash \sigma_{\theta}(T_1) = X_2 \backslash \sigma_{\theta}(T_1) = \phi_{\theta'}(X'_1), \tag{22}$$

其中 $X_2 = \phi_{\theta'_2}(X'_1)$,$T_1 = \phi_{\theta'}(\sigma_{\theta'}(X'_1)) \backslash \sigma_{\theta'}(X'_1)$,将(22)式代入到(21)式,可得到关于 X 的第二个近似式:

$$X \approx X_2 \backslash \sigma_{\theta}(T_1) \cup \sigma_{\theta}(S_1) \backslash T_0 \cup S_0. \tag{23}$$

重复同样的步骤,一般地,X_i 和 X'_i 可近似地表示为:

$$\begin{cases} X_i \approx \gamma_{\theta_{i+1}}(X_i) \cup \sigma_{\theta_i}(S_i) = X'_i \cup \sigma_{\theta_i}(S_i) = \gamma_{\theta_i}(X_i), \\ X'_i \approx \phi_{\theta'_{i+1}}(X'_i) \backslash \sigma_{\theta_i}(T_i) = X_{i+1} \backslash \sigma_{\theta_i}(T_i) = \phi_{\theta'_i}(X'_i), \\ i = 0,1,2,\cdots,n-1. \end{cases}$$

最终可得 X 的两个近似表示式:

$$\begin{aligned} X \approx\ & X_n \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \\ & \sigma_{\theta}(T_1) \cup \sigma_{\theta}(S_1) \backslash T_0 \cup S_0, \end{aligned} \tag{24}$$

及

$$\begin{aligned} X \approx\ & X'_n \cup \sigma_{\theta_n}(S_n) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \\ & \sigma_{\theta}(T_1) \cup \sigma_{\theta}(S_1) \backslash T_0 \cup S_0, \end{aligned} \tag{25}$$

若正整数 N,使得 $\varepsilon_{\theta_{N+1}}(X_N) = \varnothing$,也即 $X'_N = \gamma_{\theta_{N+1}}(X_N) = \varnothing$,则(25)式可表示为:

$$X \approx \sigma_{\theta_N}(S_N) \backslash \sigma_{\theta_{N-1}}(T_{N-1}) \cup \sigma_{\theta_{N-1}}(S_{N-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0. \quad (26)$$

4.2　近似公式的证明

可以证明,以上所得到的关于 X 的所有近似表示式均为等式.下面在4.1节相关公式推证的基础上展开证明.

对第一个近似公式(21)式,由 $X' \subseteq \phi_{\theta'}(X') = X_1$,开运算的增性[5]和(20)式知:

$$\gamma_\theta(X') \subseteq \gamma_\theta(X_1) \Leftrightarrow \gamma_\theta(X') \subseteq X'_1 \cup \sigma_\theta(S_1),$$

又 $X' = \gamma_\theta(X)$,根据开运算的等幂性[5]可得:

$$\gamma_\theta(X') \subseteq X'_1 \cup \sigma_\theta(S_1) \Leftrightarrow \gamma_\theta(\gamma_\theta(X)) \subseteq X'_1 \cup \sigma_\theta(S_1) \Leftrightarrow \gamma_\theta(X) \subseteq X'_1 \cup \sigma_\theta(S_1) \Leftrightarrow X' \subseteq X'_1 \cup \sigma_\theta(S_1),$$

由 $X' = \phi_{\theta'}(X') \backslash T_0 = X_1 \backslash T_0$ 知,X' 不含有 T_0 中任何点,故有

$$X' = X_1 \backslash T_0 \subseteq X'_1 \cup \sigma_\theta(S_1) \backslash T_0,$$

两边同时并上 S_0,利用(19)式,得:

$$X = X_1 \backslash T_0 \cup S_0 \subseteq X'_1 \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (27)$$

这说明第一个近似公式(21)式的右边包含 X 作为子集.

对于第二个近似公式(23)式,根据(22)式及闭运算的扩展性[5]可知:

$$X'_1 \subseteq \phi_{\theta'}(X'_1) \Leftrightarrow X'_1 \subseteq X_2 \backslash \sigma_\theta(T_1), \quad (28)$$

将(28)式带入到(27)式中,可得:

$$X \subseteq X_2 \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (29)$$

这说明第二个近似公式(23)式的右边包含 X 作为子集.

对于(25)式中 $n=2$ 的情形,由 $X'_1 \subseteq \phi_{\theta'_2}(X'_1) = X_2$ 及开运算的增性[5],可知:$\gamma_{\theta_2}(X'_1) \subseteq \gamma_{\theta_2}(X_2) \Leftrightarrow \gamma_{\theta_2}(X'_1) \subseteq X'_2 \cup \sigma_{\theta_2}(S_2)$,其中 $X'_1 = \gamma_{\theta_2}(X_1)$,根据开运算的等幂性[5]可知:

$$\gamma_{\theta_2}(X'_1) \subseteq X'_2 \cup \sigma_{\theta_2}(S_2) \Leftrightarrow \gamma_{\theta_2}(\gamma_{\theta_2}((X_1)) \subseteq X'_2 \cup \sigma_{\theta_2}(S_2) \Leftrightarrow \gamma_{\theta_2}(X_1) \subseteq X'_2 \cup \sigma_{\theta_2}(S_2) \Leftrightarrow X'_1 \subseteq X'_2 \cup \sigma_{\theta_2}(S_2),$$

又由 $X'_1 \subseteq X_2 \backslash \sigma_\theta(T_1)$ 可知 X'_1 不含有 $\sigma_\theta(T_1)$ 中的任何点,因而有:

$$X'_1 \subseteq X'_2 \cup \sigma_{\theta_2}(S_2) \backslash \sigma_\theta(T_1), \quad (30)$$

将(30)式代入到(27)式,可得:

$$X \subseteq X'_2 \cup \sigma_{\theta_2}(S_2) \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (31)$$

这说明当 $n=2$ 时,(25)式的右边包含 X 作为子集.

重复类似的推证,一般地可以得出(24)式及(25)式亦满足:

$$X \subseteq X_n \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (32)$$

$$X \subseteq X'_n \cup \sigma_{\theta_n}(S_n) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (33)$$

对于 $X'_n = \gamma_{\theta_{n+1}}(X_n)$,若存在正整数 N,使得 $\varepsilon_{\theta_{N+1}}(X_N) = \varnothing$,则(33)式可进一步简化为:

$$X \subseteq \sigma_{\theta_N}(S_N) \backslash \sigma_{\theta_{N-1}}(T_{N-1}) \cup \sigma_{\theta_{N-1}}(S_{N-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0. \quad (34)$$

由此证明了以上关于 X 的所有近似表示式的右边均包含 X 作为其子集.

为证明以上近似公式中的反包含关系亦成立,运用集合的补运算及开、闭运算的对偶性[5]进行相关推证.关于 X^C 可以得出:

$$X^C = (X_1)^C \cup T_0 \backslash S_0, \quad (35)$$

关于 X^C 的一般形式的近似公式如下:

$$(X)^C \approx (X_n)^C \cup \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \sigma_{\theta_{n-1}}(S_{n-1}) \cup \cdots \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (36)$$

$$(X)^C \approx (X'_n)^C \backslash \sigma_{\theta_n}(S_n) \cup \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \sigma_{\theta_{n-1}}(S_{n-1}) \cup \cdots \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (37)$$

其中 $(X_i)^C$ 和 $(X'_i)^C$ 可以近似地表示为:

$$\begin{cases} (X_i)^c \approx (\gamma_{\theta_{i+1}}(X_i))^C \backslash \sigma_{\theta_i}(S_i) = (X'_i)^C \backslash \sigma_{\theta_i}(S_i) = (\gamma_{\theta_i}(X_i))^C = \phi_{\theta'_i}((X_i)^C), \\ (X'_i)^C \approx (\phi_{\theta'_{i+1}}(X'_i))^C \cup \sigma_{\theta_i}(T_i) = (X_{i+1})^C \cup \sigma_{\theta_i}(T_i) = (\phi_{\theta'_i}(X'_i))^C = \gamma_{\theta_i}((X'_i)^C), \\ i = 1,2,3,\cdots,n. \end{cases}$$

因而,在(37)式中,当 $n=1$ 时,由闭运算的扩展性[5]可知:

$$(X_1)^C \subseteq \phi_{\theta'}((X_1)^C) = (X'_1)^C \backslash \sigma_\theta(S_1),$$

结合(35)式也即:

$$X^C \subseteq (X'_1)^C \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (38)$$

这说明当 $n=1$ 时,(37)式的右边包含 X^C 作为其子集.对于(36)式,当 $n=2$ 时,有:

$$\gamma_\theta((X_1)^C) \subseteq \gamma_\theta((X'_1)^C) = (X_2)^C \cup \sigma_\theta(T_1) \Leftrightarrow \gamma_\theta((X')^C) \subseteq \gamma_\theta((X'_1)^C) = (X_2)^C \cup \sigma_\theta(T_1) \Leftrightarrow (X_1)^C \subseteq \gamma_\theta((X'_1)^C) = (X_2)^C \cup \sigma_\theta(T_1), \quad (39)$$

将(39)式代入到(35)式中可得:$X^C \subseteq (X_2)^C \cup \sigma_\theta(T_1) \cup T_0 \backslash S_0$,(38)式表明 X^C 不含有 $\sigma_\theta(S_1)$ 中的点,故:

$$X^C \subseteq (X_2)^C \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (40)$$

这说明当 $n=2$ 时,(36)式的右边包含 X^C 作为其子集.对于(37)式,当 $n=2$ 时,根据闭运算的扩展性[5]可知:

$$(X_2)^C \subseteq \phi_{\theta'_2}((X_2)^C) = (X'_2)^C \backslash \sigma_{\theta_2}(S_2), \quad (41)$$

将(41)式代入到(40)式便可得：

$$X^C \subseteq (X'_2)^C \backslash \sigma_{\theta_2}(S_2) \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (42)$$

这说明当 $n = 2$ 时，(37)式的右边包含 X^C 作为其子集.

重复类似的推证，一般地可以得出(36)及(37)式亦满足：

$$X^C \subseteq (X_n)^C \cup \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \sigma_{\theta_{n-1}}(S_{n-1}) \cup \cdots \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (43)$$

$$X^C \subseteq (X'_n)^C \backslash \sigma_{\theta_n}(S_n) \cup \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \sigma_{\theta_{n-1}}(S_{n-1}) \cup \cdots \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0. \quad (44)$$

如果存在正整数 N，使得 $(\varepsilon_{N+1}(X_N))^C = \varnothing$，也即 $(X'_N)^C = (\gamma_{\theta_{N+1}}(X_N))^C \neq \varnothing$，则(44)式可进一步化简为：

$$X^C \subseteq \sigma_{\theta_{n-1}}(T_{n-1}) \backslash \sigma_{\theta_n}(S_n) \backslash \sigma_{\theta_{n-1}}(S_{n-1}) \cup \cdots \cup \sigma_\theta(T_1) \backslash \sigma_\theta(S_1) \cup T_0 \backslash S_0, \quad (45)$$

由此证明了 X 的近似等式的右边均是 X 的子集. 综合以上的推证结果可知，4.1 节中所有关于 X 的近似式均为等式，即一般地有：

$$X = X_n \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (46)$$

$$X = X'_n \cup \sigma_{\theta_n}(S_n) \backslash \sigma_{\theta_{n-1}}(T_{n-1}) \cup \sigma_{\theta_{n-1}}(S_{n-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (47)$$

$$X = \sigma_{\theta_N}(S_N) \backslash \sigma_{\theta_{N-1}}(T_{N-1}) \cup \sigma_{\theta_{N-1}}(S_{N-1}) \backslash \cdots \backslash \sigma_\theta(T_1) \cup \sigma_\theta(S_1) \backslash T_0 \cup S_0, \quad (48)$$

(48)式给出了内、外骨架的 SV 形态骨架变换的推广形式.

5　结束语

本文将 SV 形态学方法应用于基于内、外骨架的图像分解及重构问题的研究，用 SV 结构元替代固定结构元，建立了 SV 形态学内、外骨架变换，将骨架变换推广到更为一般的形式，弥补了固定结构元对图像重构方法的局限性. 此外，本文还研究了将内、外骨架同时作为分解成分时的图像分解和重构问题，得到了 SV 内骨架变换和 SV 外骨架变换的一个统一形式，并给出了一系列相关的推算和证明. 本文就 SV 形态学方法在图像分解问题中所涉及的相关算法的理论框架进行了研究，后续工作将对所建立的各类图像分解算法进行实验分析，以完善我们的研究工作.

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