含有3个奇异临界方程的椭圆方程组的显式基态解

康东升,马玉恒,田丹丹
(中南民族大学 数学与统计学学院,武汉 430074)

摘 要 研究了一类包含3个奇异临界方程和带有强耦合Hardy项的椭圆方程组.利用变分法,研究了相关Sobolev最佳常数的达到函数对,首次发现了椭圆方程组的一类显式基态解.

关键词 临界椭圆方程组;基态解;最佳Sobolev常数;Hardy不等式
中图分类号 O175.25 **文献标识码** A **文章编号** 1672-4321(2019)03-0472-04
DOI 10.12130/znmdzk.20190327
引用格式 康东升,马玉恒,田丹丹. 含有3个奇异临界方程的椭圆方程组的显式基态解[J].中南民族大学学报(自然科学版),2019,38(3):472-475.
KANG Dongsheng, MA Yuheng, TIAN Dandan. Explicit ground state solutions to elliptic systems involving three singular critical equations[J]. Journal of South-Central University for Nationalities(Natural Science Edition), 2019,38(3):472-475.

Explicit ground state solutions to elliptic systems involving three singular critical equations

KANG Dongsheng, MA Yuheng, TIAN Dandan
(College of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, China)

Abstract In this paper, an elliptic system is studied, which involves three singular critical equations and strongly-coupled Hardy terms. By variational methods, the minimizers of extremal functions to the related best Sobolev constant are studied and an explicit form of ground state solutions to the elliptic system is found for the first time.

Keywords critical elliptic system; ground state solution; the best Sobolev constant; Hardy inequality

1 相关知识

本文主要研究下列奇异临界椭圆方程组:

$$\begin{cases} -\Delta u - \dfrac{\lambda_{1,1}u + \lambda_{1,2}v + \lambda_{1,3}w}{|x|^2} = \\ \quad u^{2^*-1} + \dfrac{\eta\alpha}{2^*}u^{\alpha-1}v^{\beta}, \\ -\Delta v - \dfrac{\lambda_{2,1}u + \lambda_{2,2}v + \lambda_{2,3}w}{|x|^2} = \\ \quad v^{2^*-1} + \dfrac{\eta\beta}{2^*}u^{\alpha}v^{\beta-1}, \\ -\Delta w - \dfrac{\lambda_{3,1}u + \lambda_{3,2}v + \lambda_{3,3}w}{|x|^2} = w^{2^*-1}, \end{cases} \tag{1}$$

其中,$u,v,w \in D^{1,2}(\mathbb{R}^N)$,$u,v,w > 0$,$x \in \mathbb{R}^N\backslash\{0\}$,$D^{1,2}(\mathbb{R}^N)$是$C_0^\infty(\mathbb{R}^N)$关于范数$(\int_{\mathbb{R}^N}|\nabla\cdot|^2)^{1/2}$的完备化空间,并且参数满足下列条件:

(H_1) $N \geqslant 3$,$\alpha > 1$,$\beta > 1$,$\alpha+\beta = 2^*$,$\eta > 0$,$\lambda_{i,j} = \lambda_{j,i} > 0$,$0 < \lambda_{3,3} \leqslant \lambda_{2,2} \leqslant \lambda_{1,1} < \bar{\mu}$,$1 \leqslant i,j \leqslant 3$,矩阵$(\lambda_{i,j})_{3\times3}$是一个正定矩阵,并且它的特征值都属于$(0,\bar{\mu})$.

这里$2^* := \dfrac{2N}{N-2}$是Sobolev临界指数,$\bar{\mu} := \left(\dfrac{N-2}{2}\right)^2$是如下Hardy不等式中的最佳Hardy常数[1]:

收稿日期 2019-04-11
作者简介 康东升(1967-),男,教授,博士,研究方向:偏微分方程,E-mail: dongshengkang@scuec.edu.cn
基金项目 国家自然科学基金资助项目(11601530);中南民族大学研究生创新基金项目(2019sycxjj337)

$$\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}dx \leqslant \frac{1}{\bar{\mu}}\int_{\mathbb{R}^N}|\nabla u|^2 dx, \forall u \in D^{1,2}(\mathbb{R}^N).$$

在上述(H_1)成立的条件下,矩阵$(\lambda_{i,j})_{3\times 3}$是正定的,并且对所有的$(u,v,w) \in (D^{1,2}(\mathbb{R}^N))^3$都有$0 < \gamma_1 < \gamma_2 < \bar{\mu}$,同时下列结果成立:

$$\gamma_1(u^2+v^2+w^2) \leqslant Q(u,v,w) \leqslant \gamma_2(u^2+v^2+w^2), \quad (2)$$

其中γ_1,γ_2分别是矩阵$(\lambda_{i,j})_{3\times 3}$的最小特征值和最大特征值,$Q(u,v,w)$是一个二次型,定义如下:

$$Q(u,v,w):=(u,v,w)(\lambda_{i,j})_{3\times 3}(u,v,w)^{\mathrm{T}},$$
$$\forall (u,v,w) \in (D^{1,2}(\mathbb{R}^N))^3.$$

根据 Hardy,Sobolev 和 Young 不等式,我们可以定义如下最佳 Sobolev 常数[2,3]:

$$S(\mu):=\inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}dx\right)^{\frac{2}{2^*}}},\mu<\bar{\mu}. \quad (3)$$

$$S^*:=\inf_{(u,v,w)\in\mathscr{D}}\left[\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2-\frac{Q(u,v,w)}{|x|^2}\right)dx\right]\Big/\left(\int_{\mathbb{R}^N}F(u,v,w)\,dx\right)^{\frac{2}{2^*}}, \quad (4)$$

其中$\mathscr{D}:=(D^{1,2}(\mathbb{R}^N))^3\setminus\{(0,0,0)\}$,$F(u,v,w)$我们定义如下:

$$F(u,v,w):=|u|^{2^*}+|v|^{2^*}+|w|^{2^*}+\eta|u|^{\alpha}|v|^{\beta}.$$

近年来,许多学者研究和讨论了涉及到 Hardy 不等式的临界椭圆方程(参见文献[2-8])和临界椭圆方程组(参见文献[9-15]),并且取得了很多成果.在文献[3]中,Terracini 证明了最佳常数$S(\mu)$的达到函数为:

$$V_{\mu}^{\varepsilon}(x):=\varepsilon^{\frac{2-N}{2}}U_{\mu}(\varepsilon^{-1}x),\forall\varepsilon>0,\mu\in[0,\bar{\mu}), \quad (5)$$

其中$U_{\mu}(x)$是如下径向对称函数:

$$U_{\mu}(x):=\left(\frac{2N(\bar{\mu}-\mu)}{\sqrt{\bar{\mu}}}\right)^{\frac{\sqrt{\bar{\mu}}}{2}}\left(|x|^{\frac{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}+|x|^{\frac{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}\right)^{-\sqrt{\bar{\mu}}}.$$

通过对这些涉及 Hardy 不等式的临界椭圆方程和方程组的研究,很多重要问题都得到了解决.

对任意的$(u,v,w) \in (D^{1,2}(\mathbb{R}^N))^3$,方程组(1)对应的能量泛函$J(u,v,w)$如下:

$$J(u,v,w):=\frac{1}{2}\int_{\mathbb{R}^N}E(u,v,w)\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N}F(u,v,w)\,dx,$$

其中$J\in C^1((D^{1,2}(\mathbb{R}^N))^3,\mathbb{R})$,$E(u,v,w)$定义如下:

$$E(u,v,w):=|\nabla u|^2+|\nabla v|^2+|\nabla w|^2-\frac{Q(u,v,w)}{|x|^2}.$$

$(u,v,w) \in (D^{1,2}(\mathbb{R}^N))^3$是方程组(1)的一个解,如果$(u,v,w)>0$,并且满足:

$$\langle J'(u,v,w),(\varphi,\phi,\psi)\rangle=0,$$
$$\forall(\varphi,\phi,\psi)\in(D^{1,2}(\mathbb{R}^N))^3,$$

其中$J'(u,v,w)$是J在(u,v,w)的 Fréchet 导数.

(u_0,v_0,w_0)是方程组(1)的一个基态解,如果(u_0,v_0,w_0)是方程组(1)的一个解,并且在方程组(1)的所有解中取得J的最小能量值,即满足:

$$J(u_0,v_0,w_0)\leqslant J(u,v,w),$$

其中(u,v,w)是方程组(1)的任意一个解.

在本文中,主要研究方程组(1)的显式基态解和对应的最佳 Sobolev 常数的达到函数对.如果在$\mathbb{R}^N\setminus\{0\}$中有$u,v,w>0$,则称$(u,v,w)$是正的.设$\gamma_2$是矩阵$(\lambda_{i,j})_{3\times 3}$的最大特征值,作如下定义:

$$f(s_{\min},t_{\min}):=\min_{s,t\geqslant 0}f(s,t),$$

$$f(s,t):=\frac{1+s^2+t^2}{(1+s^{2^*}+t^{2^*}+\eta s^{\beta})^{\frac{2}{2^*}}},s,t\geqslant 0, \quad (6)$$

$$\mu_*:=[\lambda_{1,1}+2\lambda_{1,2}s_{\min}+2\lambda_{1,3}t_{\min}+2\lambda_{2,3}s_{\min}t_{\min}+\lambda_{2,2}s_{\min}^2+\lambda_{3,3}t_{\min}^2]/(1+s_{\min}^2+t_{\min}^2),$$

其中$(s_{\min},t_{\min})$是函数$f(s,t)$,$s,t\geqslant 0$的最小值点.

本文的主要结果归纳为如下定理.

定理 1　假设条件(H_1)成立,$V_{\mu}^{\varepsilon}(x)$是(5)式中定义的$S(\mu)$的达到函数,并且$\mu_*=\gamma_2$,则有:

$$S^*=f(s_{\min},t_{\min})S(\mu_*)=f(s_{\min},t_{\min})S(\gamma_2),$$

最佳 Sobolev 常数S^*有径向对称并且严格递减的达到函数对,形式如下:

$$\{C(V_{\gamma_2}^{\varepsilon},s_{\min}V_{\gamma_2}^{\varepsilon},t_{\min}V_{\gamma_2}^{\varepsilon}),C,\varepsilon>0\},$$

其中有一个函数对是方程组(1)的显式基态解.

在本文中,为了书写方便我们用C来表示常数,有时也会省略积分式中的 dx.

2　主要结果的证明

定理 1 的证明　假设(H_1)成立.发现函数$f(s,t)$在原点和无穷远点极限存在并且都为 1.经过计算

发现,最小值$\min\limits_{s,t\geqslant 0} f(s,t)$存在并且最小值点$(s_{\min},t_{\min})\neq(0,0)$.进一步地,我们有如下结论:

$$f'_s(s_{\min},t_{\min})=f'_t(s_{\min},t_{\min})=0,$$
$$0<f(s_{\min},t_{\min})<1.$$

对任意的$w\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$,利用函数对$(w,s_{\min}w,t_{\min}w)$在(4)式中检验Rayleigh商,可以得到:

$$S^*\leqslant \frac{\int_{\mathbb{R}^N}\left(|\nabla w|^2+|s_{\min}\nabla w|^2+|t_{\min}\nabla w|^2-\frac{Q(w,s_{\min}w,t_{\min}w)}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}F(w,s_{\min}w,t_{\min}w)\right)^{\frac{2}{2^*}}}$$
$$=f(s_{\min},t_{\min})\frac{\int_{\mathbb{R}^N}\left(|\nabla w|^2-\mu_*\frac{w^2}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}|w|^{2^*}\right)^{\frac{2}{2^*}}},$$

结合(3)式,可以得到:

$$S^*\leqslant f(s_{\min},t_{\min})S(\mu_*).\qquad(7)$$

设$\{(u_n,v_n,w_n)\}\subset\mathscr{D}$是$S^*$的极小化序列,并且定义$z_n=s_nv_n,y_n=t_nw_n$,其中:

$$s_n=\left(\left(\int_{\mathbb{R}^N}|v_n|^{2^*}\right)^{-1}\int_{\mathbb{R}^N}|u_n|^{2^*}\right)^{\frac{1}{2^*}},$$
$$t_n=\left(\left(\int_{\mathbb{R}^N}|w_n|^{2^*}\right)^{-1}\int_{\mathbb{R}^N}|u_n|^{2^*}\right)^{\frac{1}{2^*}},$$

由此可以得到:

$$\int_{\mathbb{R}^N}|z_n|^{2^*}=\int_{\mathbb{R}^N}|y_n|^{2^*}=\int_{\mathbb{R}^N}|u_n|^{2^*},\qquad(8)$$

由Young不等式和(8)式,可以得到:

$$\int_{\mathbb{R}^N}|u_n|^{\alpha}|z_n|^{\beta}\leqslant\frac{\alpha}{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*}+\frac{\beta}{2^*}\int_{\mathbb{R}^N}|z_n|^{2^*}=\int_{\mathbb{R}^N}|u_n|^{2^*},\qquad(9)$$

又由(2),(3),(8)和(9)式可得:

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2+|\nabla w_n|^2-\frac{Q(u_n,v_n,w_n)}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}F(u_n,v_n,w_n)\right)^{\frac{2}{2^*}}}\geqslant$$
$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|\nabla v_n|^2+|\nabla w_n|^2-\frac{\gamma_2(u_n^2+v_n^2+w_n^2)}{|x|^2}\right)}{\left(F(1,s_n^{-1},t_n^{-1})\int_{\mathbb{R}^N}|u_n|^{2^*}\right)^{\frac{2}{2^*}}}$$
$$\geqslant\frac{\int_{\mathbb{R}^N}\left(|\nabla u_n|^2-\gamma_2\frac{u_n^2}{|x|^2}\right)}{\left(F(1,s_n^{-1},t_n^{-1})\int_{\mathbb{R}^N}|u_n|^{2^*}\right)^{\frac{2}{2^*}}}+$$
$$\frac{s_n^{-2}\int_{\mathbb{R}^N}\left(|\nabla z_n|^2-\gamma_2\frac{z_n^2}{|x|^2}\right)}{\left(F(1,s_n^{-1},t_n^{-1})\int_{\mathbb{R}^N}|z_n|^{2^*}\right)^{\frac{2}{2^*}}}+$$
$$\frac{t_n^{-2}\int_{\mathbb{R}^N}\left(|\nabla y_n|^2-\gamma_2\frac{y_n^2}{|x|^2}\right)}{\left(F(1,s_n^{-1},t_n^{-1})\int_{\mathbb{R}^N}|y_n|^{2^*}\right)^{\frac{2}{2^*}}}\geqslant$$
$$f(s_n^{-1},t_n^{-1})S(\gamma_2)\geqslant f(s_{\min},t_{\min})S(\gamma_2),$$

当$n\to\infty$时,可以得到:

$$S^*\geqslant f(s_{\min},t_{\min})S(\gamma_2).\qquad(10)$$

假设$\mu_*=\gamma_2$,由(7)式和(10)式可以得到:

$$S^*=f(s_{\min},t_{\min})S(\gamma_2),$$

由此可以得到S^*有如下形式的达到函数对:

$$\{C(V^{\varepsilon}_{\gamma_2}(x),s_{\min}V^{\varepsilon}_{\gamma_2}(x),t_{\min}V^{\varepsilon}_{\gamma_2}(x)),C,\varepsilon>0\},$$

因为$f'_s(s_{\min},t_{\min})=f'_t(s_{\min},t_{\min})=0$,通过直接计算,可以得到存在$C^*>0$,使得方程组(1)有如下形式的显示基态解:

$$C^*(V^{\varepsilon}_{\gamma_2}(x),s_{\min}V^{\varepsilon}_{\gamma_2}(x),t_{\min}V^{\varepsilon}_{\gamma_2}(x)),\varepsilon>0,$$

也就是说,$(u_\varepsilon,v_\varepsilon,w_\varepsilon):=C^*(V^{\varepsilon}_{\gamma_2},s_{\min}V^{\varepsilon}_{\gamma_2},t_{\min}V^{\varepsilon}_{\gamma_2})$是方程组(1)的一个解,并且满足:

$$\int_{\mathbb{R}^N}E(u_\varepsilon,v_\varepsilon,w_\varepsilon)=\int_{\mathbb{R}^N}F(u_\varepsilon,v_\varepsilon,w_\varepsilon)=$$
$$(f(s_{\min},t_{\min})S(\gamma_2))^{\frac{N}{2}}=(S^*)^{\frac{N}{2}},$$

基态解$(u_\varepsilon,v_\varepsilon,w_\varepsilon)$对应的极小能量值为:

$$J(u_\varepsilon,v_\varepsilon,w_\varepsilon)=\frac{1}{2}\int_{\mathbb{R}^N}E(u_\varepsilon,v_\varepsilon,w_\varepsilon)-\frac{1}{2^*}\int_{\mathbb{R}^N}F(u_\varepsilon,v_\varepsilon,w_\varepsilon)=\frac{1}{N}(S^*)^{\frac{N}{2}}.$$

定理1证明完毕.

参 考 文 献

[1] HARDY G, LITTLEWOOD J, POLYA G. Inequalities [M]. Cambridge: Cambridge University Press, 1952: 239-243.

[2] ABDELLAOUI B, FELLI V, PERAL I. Existence and nonexistence results for quasilinear elliptic equations involving the p-laplacian with a critical potential [J]. Annali di Matematica Pura ed Applicata, 2003, 182: 247-270.

[3] TERRACINI S. On positive entire solutions to a class of

equations with a singular coefficient and critical exponent [J]. Advances in Differential Equations, 1996, 1(2): 241-264.

[4] CAO D, HAN P. Solutions to critical elliptic equations with multi-singular inverse square potentials [J]. Journal of Differential Equations, 2006, 224: 332-372.

[5] CAO D, YAN S. Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential [J]. Calculus of Variations and Partial Differential Equations, 2010, 38: 471-501.

[6] CATRINA F, WANG Z. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions [J]. Communications on Pure and Applied Mathematics, 2001, 54: 229-257.

[7] DUPAIGEN L. A nonlinear elliptic PDE with the inverse square potential [J]. Journal d'Analyse Mathématique, 2002, 86: 359-398.

[8] FELLI V, TERRACINI S. Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity [J]. Communications in Partial Differential Equations, 2006, 31: 469-495.

[9] ABDELLAOUI B, FELLI V, PERAL I. Some remarks on systems of elliptic equations doubly critical in the whole $\mathbb{R}^N$ [J]. Calculus of Variations and Partial Differential Equations, 2009, 34: 97-137.

[10] CHEN Z, ZOU W. Existence and symmetry of positive ground states for a doubly critical Schrödinger system [J]. Transactions of the American Mathematical Society, 2015, 367(5): 3599-3646.

[11] CHEN Z, ZOU W. A remark on doubly critical elliptic systems [J]. Calculus of Variations and Partial Differential Equations, 2014, 50: 939-965.

[12] KANG D, LIU X. Singularities of solutions to elliptic systems involving different Hardy type terms [J]. Journal of Mathematical Analysis and Applications, 2018, 468: 757-765.

[13] KANG D, LIU X. A note on coupled elliptic systems involving different Hardy type terms [J]. Applied Mathematics Letters, 2019, 89: 35-40.

[14] KANG D, XU L. A critical surface for the solutions to singular elliptic systems [J]. Journal of Mathematical Analysis and Applications, 2019, 472: 2017-2033.

[15] XIANG C. Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential [J]. Journal of Differential Equations, 2015, 259: 3929-3954.

(责任编辑　曹东)